Time Consistency and the Duration of Government Debt: A Signalling Theory of Quantitative Easing*

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Abstract

We present a signalling theory of quantitative easing in which open market operations that change the duration of outstanding nominal government debt affect the incentives of the central bank in determining the real interest rate. In a time consistent (Markov-perfect) equilibrium of a sticky-price model with coordinated monetary and fiscal policy, we show that shortening the duration of outstanding government debt provides an incentive to the central bank to keep short-term real interest rates low in future in order to avoid capital losses. In a liquidity trap situation then, where the current short-term nominal interest rate is up against the zero lower bound, quantitative easing can be effective to fight deflation and a negative output gap as it leads to lower real long-term interest rates by lowering future expected real short-term interest rates. We show illustrative numerical examples that suggest that the benefits of quantitative easing in a liquidity trap can be large in a way that is not fully captured by some recent empirical studies.

JEL Classification: E31, E52, E58; E61; E63

Keywords: Time-consistent equilibrium; Duration of government debt; Quantitative easing; Liquidity trap; Signalling; Zero lower bound; Sticky prices

*Preliminary. We thank Oli Coibion, Mark Gertler, Andy Levin, Emi Nakamura, seminar participants at Emory University, University of Texas at Austin, and Brown University, and conference participants at NBER ME spring meeting, HKIMR/New York Fed conference on "Domestic and International Dimensions of Unconventional Monetary Policy," Society of Economic Dynamics Annual meeting, Mid-west Macro spring meeting, CEPR European Summer Symposium in International Macroeconomics, Annual Conference on Computing and Finance, and NBER Japan Project Meeting for helpful comments and suggestions. This version: Sept 2014.

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1 Introduction

During the recent global financial crisis, central banks in many advanced economies such as the United States engaged in various forms of “unconventional” monetary policy actions as the short-term interest rate, the traditional policy instrument, was up against the zero lower bound. One form of such policy actions involved changes in the size and/or composition of the central bank’s balance sheet. In particular, the Federal Reserve in the United States carried out several large scale purchases of assets in recent years, a program often referred to collectively as Large Scale Asset Purchases (LSAPs). A considerable part of LSAPs involved buying long term government bonds, or “quantitative easing,” a naming convention we use here.

What is quantitative easing? It is when the government buys long-term government debt with money. Since the nominal interest rate was zero in recent years when this policy was implemented in the United States, it makes no difference if this was done by printing money (or more precisely bank reserves) or by issuing short-term government debt: both are government issued papers that yield a zero interest rate. From the perspective of the government as a whole, at least, quantitative easing at zero nominal interest rates can then simply be thought of as shortening the maturity of outstanding government debt.[1] The government is simply exchanging longer term liabilities in the hands of the public with shorter term ones.

The main goal of quantitative easing in the United States was to reduce long-term interest rates, even when the short-term nominal interest rate could not be reduced further, and thereby, stimulate the economy. Indeed, several empirical studies find evidence of reduction in long-term interest rates following these policy interventions by the Federal Reserve. For example, Gagnon et al (2011) estimate that the 2009 program that involved buying various types of debt worth \$1.75 trillion reduced long-term interest rates by 58 basis points while Krishnamurthy and Vissing-Jorgensen (2011) estimate that the 2010 program that involved buying long-term government debt worth \$600 billion reduced long-term interest rates by 33 basis points. In addition, Hamilton and Wu (2012), Swanson and Williams (2013), and Bauer and Rudebusch (2013) also find similar effects on long-term interest rates.[2]

From a theoretical perspective however, the effect of such policy is not obvious since open market operations of this kind are neutral (or irrelevant) in standard macroeconomic models. This was pointed out first in a well-known contribution by Wallace (1981) and further extended by Eggertsson and Woodford (2003) to a model with sticky prices and an explicit zero lower bound on nominal interest rates. These papers showed how absent some restrictions in asset trade that prevent arbitrage, a change in the relative supplies of various assets in the hands of the private sector has no effect on equilibrium quantities and asset prices in prototypical macroeconomic models.

For this reason, some papers have recently incorporated frictions such as participation constraints due to “preferred habitat” motives in order to make assets of different maturities imperfect substitutes. This in turn negates the neutrality of open market operations as in such an environment, quantitative easing can reduce long-term interest rates because it decreases the risk-premium. For example, Chen, Curdia, and Ferrero (2012) augment a quantitative sticky price model with such segmented market frictions and show that purchases of long-term bonds by the central bank can reduce long-term interest rates by decreasing the risk-premium. They nevertheless find the effects to be fairly modest: based on their estimated model,

[1] Indeed our model will be one where we will consider a consolidated government budget constraint and joint conduct of optimal monetary and tax policy.

[2] Note however that empirical studies typically measure nominal interest rates, while theoretically, it is the ability to influence real interest rates that matter. This issue is quite important and we emphasize it in detail later.

they find that a \$600 billion reduction in outstanding long-term debt, along with a commitment to keep short-term interest rates at zero for 4 quarters, increases inflation by 3 basis points (annualized) and GDP growth by 0.13% (annualized).

As Woodford (2012) argues however (building on Eggertsson and Woodford (2003)), quantitative easing need not be effective only because it reduces risk premiums. It can also reduce long-term interest rates if such policy intervention signals to the private sector that the central bank will keep the short-term interest rates low once the zero lower bound is no longer a constraint in the future. In fact, arguably, much of the findings of the empirical literature on reduction of long-term interest rates due to quantitative easing can be attributed to expectations of low future short-term interest rates. Indeed, Krishnamurthy and Vissing-Jorgensen (2011) and Bauer and Rudebusch (2013) find evidence in support of this channel in their study of the various quantitative easing programs.[3]

Our contribution is to provide a formal theoretical model of such a "signalling" role of quantitative easing in a standard general equilibrium sticky price model.[4] In particular, we consider coordinated optimal monetary and fiscal policy under discretion and show that in a Markov-perfect (time-consistent) equilibrium, shortening the duration of outstanding government debt provides an incentive to the central bank to keep the short-term real interest rate low in the future. This constitutes optimal policy under discretion because it avoids capital losses on the government's balance sheet, which if realized, would entail raising taxes that are costly in the model. The key intuition for this result is that if the government holds larger amount of short-term debt then current real short term rate directly affects the cost of rolling the debt over period by period, while the cost of rolling over long-term debt is not affected as strongly period by period (since the interest rate on that debt is predetermined at the time the policy is set). This implies that shortening the maturity of outstanding debt increases the incentive of the government to keep real rates low.

In a liquidity trap situation, where the current short-term nominal interest rate is up against the zero lower bound and the economy suffers from deflation and a large negative output gap, it is well known that signalling about future policy can be very effective. For example, Krugman (1998) emphasizes the importance of raising inflation expectations and Eggertsson and Woodford (2003) emphasize the commitment to lower future short-term nominal interest rate and allowing output to overshoot its steady state level. But are these type of commitments credible? In fact, it is well-understood that commitment to future expansionary policy at the zero lower bound is difficult due to time-inconsistency problems. While the public understands the benefits today of committing to lower future real interest rates, it also appreciates the government's incentive in the future to renege on these promises once the economy has recovered (this leads to the so called "deflation bias" developed in Eggertsson (2006)). Our main result, then, is that quantitative easing, or shortening the duration of government debt, makes promises of expansionary future policy "credible" because it provides an incentive to the central bank to keep short-term real interest rates low in future to avoid balance sheet losses. This mitigates the extent of deflation and negative output gap that would occur otherwise in the Markov-perfect equilibrium.[5]

[3] Woodford (2012) makes a similar argument regarding empirical evidence on most recent balance sheet policies by the Federal Reserve.

[4] Gagnon et al (2011) label this role played by the expected path of future short-term interest rates as a "signalling" role for quantitative easing. In the literature on central bank intervention in foreign exchange markets, this term has been used often and appears to have been first coined by Mussa (1981) to discuss how foreign exchange interventions might be used to signal future changes in monetary policy.

[5] Jeanne and Svensson (2007) assume net worth concerns on the part of the central bank directly and show how buying foreign exchange is useful during a liquidity trap as it commits the central bank to not appreciating the exchange rate in future (since doing so would entail capital losses on the central bank's balance sheet). Berriel, Bhattarai, and Mendes (2013) show that buying long-term bonds acts as a commitment device for the same reason. The main difference in this paper is the consideration of joint conduct of monetary and fiscal policy along with a welfare-theoretic loss function for the government.

In our calibrated model, we show numerical experiments in which these effects of quantitative easing can be very large. For example, with government debt maturity of 16 quarters, we calibrate the size of the negative demand shock (that makes the zero lower bound binding) such that output drops by 10% and (annualized) inflation by 2%. In such a case, reducing the duration of government debt by 7 months decreases (annualized) deflation and the negative effect on output by 95.4 and 140 basis points respectively.[6] This suggests that the signalling channel may explain all the effects of quantitative easing found in the empirical literature. In addition, we show that if the duration were to be reduced by 20.6 months, that is triple the size of our baseline experiment for quantitative easing, the negative output gap at the zero lower bound would be completely eliminated.

Before presenting the model we want to clarify an important element of our analysis. A key to understanding optimal policy in the New Keynesian model at the zero bound is that it involves committing to lower future short-term *real* interest rates.[7] The short term real interest rate is the difference between future short-term nominal interest rate and expected inflation. A lower short-term real interest rate can be achieved by either a lower future short-term nominal interest rate or higher future expected inflation. A commitment of this kind will under some parameterization of the model be achieved via higher expected inflation *and higher future short-term nominal interest rates* – even if the difference between the two (the real rate) is going down. An implication of this is that a successful policy of quantitative easing aimed at lowering long-term real interest rate may involve an *increase in expected future nominal interest rates* and therefore, an increase in long-term nominal rates at the zero lower bound. This is relevant because some empirical analyses of the effect of quantitative easing focus only on long-term *nominal interest rates*, which our analysis suggests is not a sufficient statistic. Even if long-term nominal interest rate decline very little in response to quantitative easing, or even if they increase, this does not by itself suggest that the policy is ineffective, as long as the real interest rate is declining.

Finally, while the paper connects with work related to the zero lower bound in models with nominal rigidities, it also is related to the theoretical literature on how the maturity structure of debt can be manipulated to eliminate the dynamic inconsistency found in many monetary models (in particular the "inflation bias") such as Lucas and Stokey (1983), Persson, Persson, and Svensson (1987 and 2006), and Alvarez, Kehoe, and Neumeyer (2004). While the focus of these papers is generally on policies that eliminates the incentive to inflate, the inflation bias, (and thus generally imply a lengthening of the duration of government debt relative to one-period debt), our application can be thought of as precisely the opposite, that is, the maturity structure of debt is made shorter to solve the "deflation bias."

2 The Model

2.1 Private sector

The model is a standard general equilibrium sticky-price closed economy set-up with an output cost of taxation, along the lines of Eggertsson (2006). The government conducts coordinated monetary and fiscal policy under discretion. While it may seem like a distraction to write out the fully non-linear model and define the equilibrium in that context, as we will later on analyze a linear quadratic version of this model,

[6] We use these numbers for illustration based on estimates in Chadha, Turner, and Zampoli (2013), which suggest that the average maturity of treasury debt held outside the Federal Reserve was around 4 years in the last 10 years and that recent Federal Reserve balance sheet policies reduced the maturity by around 7 months.

[7] Werning (2012) makes a related point regarding the ultimate goal of optimal policy as one of generating an output boom once the trap is over.

this is useful for two reasons. First, we will formally show that the linearized first-order conditions of the government's original non-linear problem are the same as in our linear quadratic model (and this in general need not be the case, see e.g. Eggertsson (2006)). Second, the non-linear version of the problem will be important once we allow for fully time-varying duration of government debt, where the linear quadratic approximation is no longer valid.

The main difference in the model from the literature is the introduction of long-term government debt. A representative household maximizes expected discounted utility over the infinite horizon

$$E_t \sum_{t=0}^{\infty} \beta^t U_t = E_t \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t\right) + g\left(G_t\right) - v(h_t) \right] \xi_t \tag{1}$$

where β is the discount factor, C_t is household consumption of the composite final good, G_t is government consumption of the composite final good, h_t is quantity supplied of labor of type i, and ξ_t is a shock. E_t is the mathematical expectation operator conditional on period-t information, $u\left(.\right)$ is concave and strictly increasing in C_t, $g\left(.\right)$ is concave and strictly increasing in G_t, and $v\left(.\right)$ is increasing and convex in h_t.[8]

The composite final good is an aggregate of a continuum of varieties indexed by i, $C_t = \int_0^1 \left[c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ is the elasticity of substitution among the varieties. The optimal price index for the composite final good is given by $P_t = \left[\int_0^1 p_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$, where $p_t(i)$ is the price of the variety i. The demand for the individual varieties is then given by $\frac{c_t(i)}{C_t} = \left(\frac{p_t(i)}{P_t} \right)^{-\varepsilon}$. Finally, G_t is defined analogously to C_t and so we omit detailed description of government spending.

The household is subject to a sequence of flow budget constraints

$$P_t C_t + B_t^S + S_t B_t + E_t \{ Q_{t,t+1} A_{t+1} \} \leq n_t h_t + \left(1 + i_{t-1}\right) B_{t-1}^S + \left(1 + \rho S_t\right) B_{t-1} + A_t - P_t T_t + \int_0^1 Z_t(i) di \tag{2}$$

where n_t is nominal wage, $Z_t(i)$ is nominal profit of firm i, B_t^S is the household's holding of one-period risk-less nominal government bond at the beginning of period $t+1$, B_t is a perpetuity bond, S_t its price, and ρ its decay factor (further described below). A_{t+1} is the value of the complete set of state-contingent securities at the beginning of period $t+1$ and $Q_{t,t+1}$ is the stochastic discount factor between periods t and $t+1$ that is used to value random nominal income in period $t+1$ in monetary units at date t.[9] Finally, i_{t-1} is the nominal interest rate on government bond holdings at the beginning of period t and T_t is government taxes.

The way we introduce long term bonds into the model is to assume that government debt not only takes the form of a one period riskfree debt, B_t^S, but that the government also issues a perpetuity in period t which pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$.[10] S_t is the price of the perpetuity nominal bond which depends on the decay factor ρ. The main convenience of introducing long term bond in this way is that we can consider government debt of arbitrary duration. For example, a value of $\rho = 0$ implies that this bond is simply a short-term bond while $\rho = 1$ corresponds to a classic console bond. More generally, in an environment with stable prices, the duration of this bond is $(1 - \beta\rho)^{-1}$. Thus, this simple assumption allows us to explore a change in the duration of government debt in a transparent way. The appendix contains details on why the budget constraint takes the form (2). In particular, the modelling of long-term bond in this way admits a simple recursive formulation of the price of old government bonds.

[8] We abstract from money in the model and are thus directly considering the "cash-less limit."

[9] The household is subject to a standard no-Ponzi game condition.

[10] We follow Woodford (2001).

For now, observe that we treat ρ as a constant. We will explore a one-time reduction in this duration as a main "comparative static" of interest. In other words, a reduction in ρ answers the question: What does a permanent reduction in the maturity of government debt do? Toward the end of the paper, however, we will extend the analysis so that ρ becomes a time varying choice variable ρ_t. The main reason for our initial benchmark assumption is simplicity (and the fact that we get a clean comparative static). But perhaps more importantly, we will see later that a one-time reduction in ρ (in a liquidity trap) turns out to be a reasonably good approximation in our model because ρ_t is close to a random walk under optimal policy under discretion at a positive interest rate.

The maximization problem of the household is now entirely standard, with the additional feature of the portfolio choice between long and short term bonds.[11] Let us now turn to the firm side of the model.

There is a continuum of monopolistically competitive firms indexed by i. Each firm produces a variety i according to the production function that is linear in labor $y_t(i) = h_t(i)$. As in Rotemberg (1983), firms face a cost of changing prices given by $d\left(\frac{p(i)}{p_{t-1}(i)}\right)$. This adjustment cost makes the firm's pricing problem dynamic. The demand function for variety i is given by

$$\frac{y_t(i)}{Y_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\varepsilon} \tag{3}$$

where Y_t is total demand for goods. The firm maximizes expected discounted profits

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s} Z_{t+s}(i) \tag{4}$$

where the period profits $Z_t(i)$ are given by

$$Z_t(i) = \left[(1+s)\, Y_t p_t(i)^{1-\varepsilon} P_t^{\varepsilon} - n_t(i) Y_t p_t(i)^{-\varepsilon} P_t^{\varepsilon} - d\left(\frac{p_t(i)}{p_{t-1}(i)}\right) P_t \right]$$

where s is a production subsidy which we will set to eliminate the steady state distortion of monopolistic competition as is common in the literature.[12]

We can now write down the necessary conditions for equilibrium that arise from the maximization problems of the private sector described above. We focus on a symmetric equilibria where all firms charge the same price and produce the same amount of output. Note that these conditions hold for any government policy.

The households optimality conditions are given by

$$\frac{v_h(h_t)}{u_C(C_t)} = \frac{n_t}{P_t} \tag{5}$$

$$\frac{1}{1+i_t} = E_t \left[\beta \frac{u_C(C_{t+1})\xi_{t+1}}{u_C(C_t)\xi_t} \Pi_{t+1}^{-1} \right] \tag{6}$$

$$S_t = E_t \left[\beta \frac{u_C(C_{t+1})\xi_{t+1}}{u_C C_t)\xi_t} \Pi_{t+1}^{-1} \left(1 + \rho S_{t+1}\right) \right] \tag{7}$$

[11] The problem of the household is thus to choose $\{C_{t+s}, h_{t+s}(i), B^S_{t+s}, B_{t+s}, A_{t+s}\}$ to maximize (1) subject to a sequence of flow budget constraints given by (2), while taking as exogenously given initial wealth and $\{P_{t+s}, n_{t+s}(i), i_{t+s}, S_{t+s}(\rho), Q_{t,t+s}, \xi_{t+s}, Z_{t+s}(i), T_{t+s}\}$.

[12] The problem of the firm is thus to choose $\{p_{t+s}(i)\}$ to maximize (4), while taking as exogenously given $\{P_{t+s}, Y_{t+s}, n_{t+s}(i), Q_{t,t+s}, \xi_{t+s}\}$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ is gross inflation.[13] The firm's optimality condition from price-setting is given by

$$\varepsilon Y_t \left[u_C\left(C_t\right) - v_y(Y_t)\right]\xi_t + u_C(C_t)\xi_t d'\left(\Pi_t\right)\Pi_t = E_t\left[\beta u_C C_{t+1})\xi_{t+1} d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] \tag{8}$$

where with some abuse of notion we have replaced v_h with v_y since in a symmetric equilibrium $h_t(i) = y_t(i) = Y_t$.

2.2 Government

There is an output cost of taxation (for example, as in Barro (1979)) captured by the function $s(T_t - T)$ where T is the steady-state level of taxes. Thus, in steady-state, there is no tax cost. Total government spending is then given by

$$F_t = G_t + s(T_t - T)$$

where G_t is aggregate government consumption of the composite final good defined before.

It remains to write down the (consolidated) flow budget constraint of the government. Note that the government issues both a one-period bond B_t^S and the perpetuity B_t. We can write the flow budget constraint as

$$B_t^S + S_t B_t = \left(1 + i_{t-1}\right) B_{t-1} + \left(1 + \rho S_t\right) B_{t-1} + P_t\left(F_t - T_t\right).$$

Next, we assume that the one-period bond is in net-zero supply (i.e. $B_t^S = 0$, which makes clear that we only introduce this bond explicitly as the risk free short term nominal rate is the key policy instrument of monetary policy), and write the budget constraint in real terms as

$$S_t b_t = \left(1 + \rho S_t\right) b_{t-1}\Pi_t^{-1} + \left(F_t - T_t\right) \tag{9}$$

where $b_t = \frac{B_t}{P_t}$. We now define fiscal policy as the choice of T_t, F_t , and b_t. For simplicity, we will from now on suppose that total government spending is constant so that $F_t = F$. Monetary policy is the choice of i_t. We simply impose the zero bound constraint on the setting of monetary policy so that[14]

$$i_t \geq 0. \tag{10}$$

2.3 Private sector equilibrium

The goods market clearing condition gives the overall resource constraint as

$$Y_t = C_t + F_t + d\left(\Pi_t\right). \tag{11}$$

We can then define the private sector equilibrium, that is the set of possible equilibria that are consistent with household and firm maximization and the technological constraints of the model. A private sector equilibrium is a collection of stochastic processes $\{Y_{t+s},\ C_{t+s},\ b_{t+s},\ S_{t+s},\ \Pi_{t+s},\ i_{t+s},\ Q_{t,t+s},\ T_{t+s},\ F_{t+s},\ G_{t+s}\}$ for $s \geq 0$ that satisfy equations (5)-(7), (8), (9), and (11), for each $s \geq 0$, given b_{t-1} and an exogenous stochastic process for $\{\xi_{t+s}\}$. To determine the set of possible equilibria in the model, we now need to be explicit about how policy is determined.

[13] We may also append a standard transversality condition as a part of these conditions or a natural borrowing limit.

[14] This bound can be explicitly derived in a variety of environment, see e.g. Eggertsson and Woodford (2003).

3 Markov-perfect Equilibrium

We characterize a Markov-perfect (time-consistent) Equilibrium in which the government cannot commit and acts with discretion every period.[15] In particular, we consider coordinated monetary and fiscal policy, where the central bank and the treasury conduct optimal monetary and fiscal policy under discretion. A key assumption in a Markov-perfect Equilibrium is that government policy cannot commit to the actions for the future government. Following Lucas and Stokey (1983), however, we suppose that the government is able to commit to paying back the nominal value of its debt.[16] The only way the government can influence future governments, then, is via any endogenous state variables that may enter the private sector equilibrium conditions. Before writing up the problem of the government, it is therefore necessary to write the system in a way that makes clear what are the endogenous state variables of the game we study.

Define the expectation variables f_t^E, g_t^E, and h_t^E. The necessary and sufficient condition for a private sector equilibrium is now that the variables $\{Y_t, C_t, b_t, S_t, \Pi_t, i_t, T_t\}$ satisfy: (a) the following conditions

$$S_t(\rho)b_t = (1 + \rho S_t(\rho))\, b_{t-1}\Pi_t^{-1} + (F - T_t) \tag{12}$$

$$1 + i_t = \frac{u_C\left(C_t\right)\xi_t}{\beta f_t^E},\ i_t \geq 0 \tag{13}$$

$$S_t(\rho) = \frac{1}{u_C\left(C_t\right)\xi_t}\beta g_t^E \tag{14}$$

$$\beta h_t^E = \varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}u_C\left(C_t\right)\xi_t - \tilde{v}_y\left(Y_t\right)\xi_t\right] + u_C\left(C_t\right)\xi_t d'\left(\Pi_t\right)\Pi_t \tag{15}$$

$$Y_t = C_t + F + d\left(\Pi_t\right) \tag{16}$$

given b_{t-1} and the expectations f_t^E, g_t^E, and h_t^E; (b) expectations are rational so that

$$f_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}\Pi_{t+1}^{-1}\right] \tag{17}$$

$$g_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] \tag{18}$$

$$h_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right]. \tag{19}$$

Note that the possible private sector equilibrium defined above depends only on the endogenous state variable b_{t-1} and shocks ξ_t. Given that the government cannot commit to future policy (apart from through the endogenous state variable), a Markov-perfect Equilibrium then requires that the expectations f_t^E, g_t^E,and h_t^E are only a function of these two state variables, i.e, we can define the expectation functions

$$f_t^E = \bar{f}^E(b_t, \xi_t) \tag{20}$$

$$g_t^E = \bar{g}^E(b_t, \xi_t) \tag{21}$$

$$h_t^E = \bar{h}^E(b_t, \xi_t). \tag{22}$$

We can now write the discretionary government's optimization problem as a dynamic programming problem

$$V(b_{t-1}, \xi_t) = \max_{i_t, T_t}[U\left(.\right) + \beta E_t V(b_t, \xi_{t+1})] \tag{23}$$

[15] See Maskin and Tirole (2001) for a formal definition of the Markov-perfect equilibrium.
[16] One could model this more explicitly by assuming that the cost of outright default is arbitrarily high.

subject to the private sector equilibrium conditions (12)-(16) and the expectation functions (20)-(22). Note that in equilibrium, the expectation functions satisfy the rational expectation restrictions (17)-(19). Here, $U(.)$ is the utility function of the household in (1) and $V(.)$ is the value function.[17] The detailed formulation of this maximization problem and the associated first-order necessary conditions, as well as their linear approximation, are provided in the appendix.[18]

4 Linear-quadratic approach

Rather than studying the fully non-linear policy problem, we will instead do a linear-quadratic approximation of the policy problem. The main reason for this approach is that it clarifies the interpretation of our result, and connects more closely to some of the earlier literature. As we will see shortly (Proposition 1), however, this approach yields the same solution as if we would have approximated directly the non-linear first-order conditions of the problem (23). When moving to time-varying ρ, however, a linear quadratic approximation will no longer be valid, in which case we will only rely on the fully non-linear statement of the government's decision problem.

We approximate our model around an efficient non-stochastic steady-state with zero inflation.[19] Moreover, there are no tax collection costs in steady-state[20]. Thus, there is a non-zero steady-state level of debt.[21] In steady-state, we assume that there is some fixed total market-value of public debt $Sb = \bar{\Gamma}$. Then, the following relationships hold

$$1 + i = \beta^{-1},\ S = \frac{\beta}{1-\rho\beta},\ b = \frac{1-\rho\beta}{\beta}\bar{\Gamma} \text{ and } T = F + \frac{1-\beta}{\beta}\bar{\Gamma}.$$

We log-linearize the private sector equilibrium conditions around the steady state above to get the relationships

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t\pi_{t+1} - r_t^e) \tag{24}$$

$$\pi_t = \kappa\hat{Y}_t + \beta E_t\pi_{t+1} \tag{25}$$

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t \tag{26}$$

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1} \tag{27}$$

where κ and σ are a function of structural model parameters that do not depend upon ρ and r_t^e is the efficient rate of interest that is a function of the shock ξ_t.[22] The coefficient $\psi \equiv \frac{T}{\bar{\Gamma}}$ is also independent of ρ in our experiment.[23]

Here, (24) is the linearized household Euler equation, (25) is the linearized Phillips curve, (26) is the linearized government budget constraint, and (27) is the linearized forward-looking asset-pricing condition.[24]

[17] Using compact notation, note that we can write the utility function as $[u(C_t) + g(F - s(T_t - T)) - v(Y_t)]\xi_t$.

[18] Note here that we assume that the government and the private-sector move simultaneously.

[19] Variables without a t subscript denote a variable in steady state. Note that output is going to be at the efficient level in steady state because of the assumption of the production subsidy (appropriately chosen) we have made before.

[20] We can think of this as being due to a limited set of lump sum taxation.

[21] The steady-state is efficient even with non-zero steady-state debt because of our assumption that taxes do not entail output loss in steady-state.

[22] The details of the derivation are in the appendix.

[23] Since we are thinking of changes in ρ in our experiment as exchanging short bonds with long bonds – effectively reducing/increasing ρ – this interpretation would imply that total value of debt in steady-state – $\bar{\Gamma}$– should remain unchanged.

[24] Variables with hats denote log-deviations from steady state except for the nominal interest rate, which is given as $\hat{\imath}_t = \frac{i_t - i}{1+i}$. Since in the non-stochastic steady state with zero inflation, $1+i = \frac{1}{\beta}$, this means that the zero lower bound on nominal interest

(24) and (25) are standard relationships depicting how current output depends on expected future output and the current real interest rate gap and how current inflation depends on expected future inflation and the current output respectively.[25]

(26) shows that since debt is nominal, its real value is decreased by inflation. Higher taxes also reduce the debt burden. Moreover, an increase in the price of the perpetuity bond decreases the real value of debt, with the effect depending on the duration of debt: longer the duration, lower is the effect of the bond price on debt. Finally, (27) shows that the price of the perpetuity bond is determined by (the negative of) expected present value of future short-term interest rates. Hence, lower current or future short-term nominal interest rate will increase the price of the perpetuity bond. Note that when $\rho = 0$, all debt is of one-period duration and (26) reduces to the standard linearized government budget constraint while (27) reduces to $\hat{S}_t = -\hat{\imath}_t$.[26]

A second-order approximation of household utility around the efficient non-stochastic steady state gives

$$U_t = -\left[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2\right] \tag{28}$$

where λ_π and λ_T are a function of structural model parameters.[27] Compared to the standard loss-function in models with sticky prices that contains inflation and output, (28) features losses that arise from output costs of taxation outside of steady-state.

To analyze optimal policy under discretion in the linear-quadratic framework we once again maximize utility, subject the now linear private sector equilibrium conditions, taking into account that the expectation are functions of the state variables of the game. In the linear system, the exogenous state is now summarized with r_t^e while the endogenous state variable is once again $\hat{b}_{t-1}$. Moreover, the expectation variables appearing in the system are now $E_t\hat{Y}_{t+1}$, $E_t\hat{S}_{t+1}$, and $E_t\pi_{t+1}$. Accordingly, we will define the game in terms of the state variables $(r_t^e, \hat{b}_{t-1})$ and the government now takes as given the expectation functions $\bar{Y}^E(\hat{b}_t, r_t^e)$, $\bar{S}^E(\hat{b}_t, r_t^e)$, and $\bar{\pi}^E(\hat{b}_t, r_t^e)$.

The discretionary government's optimization problem can then be written recursively as a linear-quadratic dynamic programming problem

$$V(\hat{b}_{t-1}, r_t^e) = \min[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2 + \beta E_t V(\hat{b}_t, r_{t+1}^e)]$$

s.t.

$$\hat{Y}_t = \bar{Y}^E(\hat{b}_t, r_t^e) - \sigma(\hat{\imath}_t - \bar{\pi}^E(\hat{b}_t, r_t^e) - r_t^e)$$

$$\pi_t = \kappa \hat{Y}_t + \beta \bar{\pi}^E(\hat{b}_t, r_t^e)$$

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi \hat{T}_t$$

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta \bar{S}^E(\hat{b}_t, r_t^e)$$

Observe that once again, in equilibrium, the expectation functions need to satisfy the rational expectations restrictions that $E_t\hat{Y}_{t+1} = \bar{Y}^E(\hat{b}_t, r_t^e)$, $E_t S_{t+1} = \bar{S}^E(\hat{b}_t, r_t^e)$, and $E_t\pi_{t+1} = \bar{\pi}^E(\hat{b}_t, r_t^e)$.

We prove in the proposition below that this linear-quadratic approach gives identical linear optimality

rates imposes the following bound on $\hat{\imath}_t : \hat{\imath}_t \geq -(1-\beta)$.

[25] We write directly in terms of output rather than the output gap since we will not be considering shocks that perturb the efficient level of output in the model.

[26] It is important to point out one techical detail in this case. The interpretation in this case of $\hat{b}_t$ is that it is the real value of the debt inclusive of the interest rate payment to be paid next period, that is, if all debt were one period $b_t = (1+i_t)\frac{B_t^S}{P_t}$.

[27] The details of the derivation are in the appendix. In particular, $\lambda_\pi = \frac{\varepsilon}{k}$.

conditions as the one obtained by linearizing the non-linear optimality conditions of the original non-linear government maximization problem that we described above. This provides the formal justification of our simplified approach.

Proposition 1 *The linearized dynamic system of the non-linear Markov Perfect Equilibrium is equivalent to the linear dynamic system of the linear-quadratic Markov Perfect Equilibrium.*

Proof. In Appendix. ■

4.1 Solution at positive interest rates

The complication in solving a Markov-perfect Equilibrium is that we do not know the unknown expectation functions $\bar{\pi}^E$, $\bar{Y}^E$, and $\bar{S}^E$. To solve this, we use the method of undetermined coefficients. Provided that the expectation functions are differentiable, the solution of the model is of the form

$$\begin{aligned} &\pi_t = \pi_b \hat{b}_{t-1} + \pi_r r_t^e,\ \hat{Y}_t = Y_b \hat{b}_{t-1} + Y_r r_t^e,\ \hat{S}_t = S_b \hat{b}_{t-1} + S_r r_t^e, \\ &\hat{\imath}_t = i_b \hat{b}_{t-1} + i_r r_t^e,\ \hat{T}_t = T_b \hat{b}_{t-1} + T_r r_t^e,\ \text{and } \hat{b}_t = b_b \hat{b}_{t-1} + b_r r_t^e \end{aligned} \tag{29}$$

where π_b, Y_b, S_b, i_b, b_b, T_b, π_r, Y_r, S_r, i_r, b_r, and T_r are unknown coefficients to be determined. We make the assumption that the exogenous process r_t^e satisfies $E_t r_{t+1}^e = \rho_r r_t^e$ where $0 < \rho_r < 1$. Then (29) implies that the expectations are given by

$$E_t \pi_{t+1} = \pi_b \hat{b}_t + \pi_r \rho_r r_t^e,\ E_t \hat{Y}_{t+1} = Y_b \hat{b}_t + Y_r \rho_r r_t^e,\ \text{and } E_t \hat{S}_{t+1} = S_b \hat{b}_t + S_r \rho_r r_t^e. \tag{30}$$

We can then formulate the Lagrangian of the government problem. We substitute out for the expectation function using (30) and suppress the shock for simplicity

$$\begin{aligned} L_t = &\frac{1}{2}(\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2) + \beta E_t V(\hat{b}_t, r_{t+1}^e) \\ &+ \phi_{1t}[\hat{Y}_t - Y_b \hat{b}_t + \sigma \hat{\imath}_t - \sigma \pi_b \hat{b}_t] + \phi_{2t}[\pi_t - \kappa \hat{Y}_t - \beta \pi_b \hat{b}_t] \\ &+ \phi_{3t}[\hat{b}_t - \beta^{-1} \hat{b}_{t-1} + \beta^{-1} \pi_t + (1-\rho)\hat{S}_t + \psi \hat{T}_t] + \phi_{4t}[\hat{S}_t + \hat{\imath}_t - \rho \beta S_b \hat{b}_t]. \end{aligned}$$

For now, we are not analyzing the effects of the shock, and hence not carrying around π_r, Y_r, S_r, i_r, b_r, and T_r since our key area of interest at this state is not the effect of the shock at positive interest rates. Instead, we will start focusing on the shock once the zero bound becomes binding.

The associated first order necessary conditions of the Lagrangian problem above and the envelope condition of the minimization problem of the government above are provided in the appendix. Our first substantiative result is that the equilibrium conditions can be simplified, in particular by eliminating the Lagrange multipliers $\phi_{1t} - \phi_{4t}$, to get

$$\lambda_\pi \pi_t + \kappa^{-1} \hat{Y}_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}] \frac{1}{\psi} \lambda_T \hat{T}_t \tag{31}$$

$$[1 - \beta \pi_b \kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma \pi_b)(1-\rho)\sigma^{-1} + \rho \beta S_b (1-\rho)] \hat{T}_t = -(\psi) \lambda_T^{-1} \beta \pi_b \kappa^{-1} \hat{Y}_t + E_t \hat{T}_{t+1} \tag{32}$$

which along with (24)-(27) define the equilibrium in the approximated economy. The final step to computing the solution is then to plug in the conjectured solution and to match coefficients on various variables in (24)-

(27), (31), and (32), along with the requirement that expectation are rational, to determine π_b, Y_b, S_b, i_b, b_b, and T_b. The details of this step are in the appendix.

The most important relationships emerging from our analysis are captured by (31) and (32). (31) is the so called "targeting rule" of our model. That represents the equilibrium (static) relationship among the three target variables π_t, $\hat{Y}_t$, and $\hat{T}_t$ that emerges from the optimization problem of the government. (31) thus captures how target variables are related in equilibrium as governed both by the weights they are assigned in the loss function (λ_π and λ_T) as well as the trade-offs among them as given by the private sector equilibrium conditions (κ^{-1} and $[\kappa^{-1}(1-\rho)\sigma^{-1}+\beta^{-1}]\frac{1}{\psi}$). Note in particular that κ^{-1} represents the trade-off between π_t and $\hat{Y}_t$ as given by (25) while $[\kappa^{-1}(1-\rho)\sigma^{-1}+\beta^{-1}]\frac{1}{\psi}$ represents the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$ as given by the combination of (24), (25), and (26).

(32) is another optimality condition characterizing the Markov-perfect equilibrium and represents the "tax-smoothing objective" of the government. In contrast to similar expressions following the work of Barro (1979), which would lead to taxes being a martingale, output appears in (32) because of sticky-prices, which makes output endogenous. Finally, because of the dynamic nature of (32), as opposed to (31), the unknown coefficients that are critical for expectations of variables, π_b, Y_b, and S_b, appear in (32). As we shall see – and this is again in contrast to the classic tax smoothing result in which debt is a random walk – the government will in general have an incentive to pay down public debt if it is above steady-state due to strategic reasons.

4.1.1 Debt dynamics, inflation, and interest rates

Let us first consider the most basic exercise to clarify the logic of the government's problem. How do the dynamics of the model look like in the absence of shocks when the only difference from steady state is that there is some initial value of debt with some fixed value of debt duration? To do this exercise we calibrate the model as follows. The parameter values we pick are given in Table 1. Following Eggertsson (2006), we choose our baseline parameters as follows: $\beta = 0.99$, $\sigma = 1$, $\kappa = 0.02$, and $\varepsilon = 8$. For the steady-state level of debt-to-taxes, $\frac{bS}{T} = \frac{\bar{\Gamma}}{T}$, we use data from the Federal Reserve Bank of Dallas to get the long-run average of market value of debt over output ($\frac{bS}{Y}$) and NIPA data to estimate the ratio of taxes over output ($\frac{T}{Y}$). This gives us the value $\frac{bS}{T} = \frac{\bar{\Gamma}}{T} = 7.2$. There is no direct counterpart to $\lambda_T = \frac{s''}{\phi+\sigma^{-1}}$ in the data, and hence we will experiment with several values and discuss our choices further below. For now, we set it at 0.8, so that tax distortions obtain less weight than output in the objective of the government. The parameter ρ is chosen to get a (baseline) duration of debt of 16 quarters. We pick this value based on estimates in Chadha, Turner, and Zampoli (2013) which suggest that the average maturity of treasury debt held outside the Federal Reserve was around 4 years in the last 10 years

Figure (1) shows the dynamics of the endogenous variables in the model for an initial value of debt that is 30 percent above the steady state debt in the model.[28] We see that if debt is above steady state, it is paid over time back to steady state. For our baseline duration of 16 quarters (solid line), the half-life of debt repayment is about 30 quarters. Note that in this transition for this parameterization, inflation is about 1 percent above steady state. Importantly, in the transition phase, we see that the real interest rate is below its steady state. As a consequence, output is also above its steady state value. Note that this result is in contrast to the classic Barro tax smoothing result whereby debt follows a random walk. The reason for this is that debt creates an incentive to create inflation for a discretionary government as further

[28] This is consistent with data from the Federal Reserve Bank of Dallas which suggests that at 2008:IV, government debt was around 30 percent above the steady state.

described below. By paying down debt back to steady state, the government thus eliminates this incentive and achieves better outcomes.

The figure illustrates that for a given maturity of government debt, debt is inflationary and implies lower future real interest rate until a new steady state is reached. What is the logic for this result? Perhaps the best way to understand the logic is by inspecting the government budget constraint (26). Recall that debt issued in nominal terms, although in the budget constraint we have rewritten it in terms of $\hat{b}_t = \frac{\frac{B_t}{P_t} - \bar{b}}{\bar{b}}$. This implies that for a given outstanding debt $\hat{b}_{t-1}$, any actual inflation will reduce the real value of the outstanding debt. Accordingly we have the term $\beta^{-1}\pi_t$ term in the budget constraint which reflects this inflation incentive. As the literature has stressed in the past (see e.g. Calvo and Guidotti (1990 and 1992)), if prices are flexible then this will reduce actual debt in equilibrium only if the inflation is unanticipated. The reason for this is that otherwise anticipated inflation will be reflected one-to-one in the interest rate paid on the debt.

Apart from the inventive to depreciate the real value of the deb via inflation, there is a second force at work in our model. In our model, the government is not only able to affect the price level, it can also have an effect on the real interest rate. Hence we see that in Figure 1 the real interest rate is below steady state during the entire transition path back to steady state. This will reduce the real interest rate payment the government needs to pay on debt – in contrast to the classic literature with flexible prices where the (ex-ante) real interest rate is exogenous. Intuitively, it may be most straight forward to see this force at work by simplifying the model down to the case in which $\rho = 0$. In that case, the budget constraint of the government can be written as

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + \hat{\imath}_t - \psi\hat{T}_t$$

and now $\hat{b}_t$ is the real value of one period risk-free nominal debt in period t which is *inclusive of interest paid* (to relate to our prevision notation in 2 then when $\rho = 0$ we have $b_t = \frac{B_t}{P_t} = \frac{(1+i_t)B_t^S}{P_t}$ where B_t^S was the one period government debt that did not include interest payment). This expression makes clear that while π_t has a direct effect by depreciating the real value of government debt, the government has another important margin by which it can influence its debt burden. The term $\hat{\imath}_t$ reflects the cost or rolling-over-cost of the one-period debt. In particular, we see that if the interest rate is low, then the cost of rolling over debt is smaller. This latter mechanism will be critical when considering the effects of varying debt maturity since its force is highly depending on the value of ρ.

4.1.2 The role of duration of debt

We now consider how reducing the value of ρ affects the transition dynamics from last section. As noted before, our main interest for this is that a natural interpretation of quantitative easing is that it corresponds to a reduction in ρ as in our model, as the duration of debt is given by $(1-\beta\rho)^{-1}$. As Figure (1) reveals, where we consider a reduction in duration from 4 years to 3.42 and 2 years (dotted and dashed lines respectively), this increases inflation in equilibrium considerably, but also reduce the real rate further. Similarly, we see that the debt is now paid down at a faster clip, a point we will return to.

To obtain some intuition for this result, let us again write out the budget constraint, this time substituting out for S_t to obtain

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + (1-\rho)[\hat{\imath}_t - \rho\beta\hat{S}^E(\hat{b}_t, r_t^e)] - \psi\hat{T}_t \tag{33}$$

We observe here that the rollover interest rate is now multiplied by the term $(1-\rho)$. Intuitively then, if a larger part of government debt is held with long maturity, the short-term rollover rate matters less, as the term of the loans are to a greater extent predetermined. Hence, the incentive of the government to lower the short-term interest rate are reduced.

4.1.3 Real interest rate incentives and duration of outstanding debt

How does the duration of debt affect the real interest rate incentives of the central bank? The effect on the real interest rate of lower duration is at the heart of the matter because what influences output is eventually the real interest rate and the aim in a zero lower bound situation is precisely to be able to decrease the short-term real interest rate (today and in future) even when the current short-term nominal interest rate is stuck at zero. That is, we are interested in the properties of

$$\hat{r}_t = \hat{\imath}_t - E_t \pi_{t+1} = (i_b - \pi_b b_b)\, \hat{b}_{t-1} = r_b \hat{b}_{t-1}$$

where $\hat{r}_t$ is the short-term real interest rate. This is important because in a liquidity trap situation, as is well-known, decreasing future real interest rate is key to mitigating negative effects on output, and so, if r_b depends negatively on duration, then we are able to provide a theoretical rationale for quantitative easing actions by the government. To clarify what is going on in the model, we first find it helpful to consider two special cases.

a) Fully flexible prices When prices are fully flexible, then as is well-known, monetary policy cannot control the (ex-ante) real interest rate $\hat{r}_t$. Then, the only way monetary policy can affect the economy is through surprise inflation as debt is nominal. In fact there is a well-known literature that addresses the issue of how the duration of nominal debt in a flexible price environment affects allocations under time consistent optimal monetary policy. For example, Calvo and Guidotti (1990 and 1992) address optimal maturity of nominal government debt in a flexible price environment while Sims (2013) explores how the response of inflation to fiscal shocks depends on the maturity of government debt under optimal monetary policy. We conduct a complementary exercise here and want to characterize how inflation incentives depend on the duration of debt under optimal monetary and fiscal policy under discretion. Thus, we are interested in how π_b depends on duration of debt.

For this exercise, we can think of this special case of fully flexible prices as $\kappa \to \infty$. Note however, that from the two optimality conditions (31) and (32), while under flexible prices $T_b = 0$, there is indeterminacy in terms of inflation and nominal interest rate dynamics.[29] This is a well-known result in monetary economics under discretion in a flexible price environment, and comes about because the government cannot affect output and taxes can be put to zero with various combinations of inflation and interest rate choices. To show our result on the role of the duration of debt, we follow the literature such as Calvo and Guidotti (1990 and 1992) and Sims (2013) and include a (very small) aggregate social cost of inflation that is independent of the level of price stickiness. Then, the objective of the government under flexible prices will be given by

$$U_t = -\left[\lambda_\pi^{'} \pi_t^2 + \lambda_T \hat{T}_t^2\right]$$

where $\lambda_\pi^{'}$ parameterizes the cost of inflation that is independent of sticky prices. Using $\lambda_\pi^{'} = 0.001$ and the rest of the parameter values from Table 1, Fig (2) shows how π_b depends on duration of debt (quarters). We see clearly that with shorter duration, there is more of an incentive to use current inflation. The intuition for

[29] Note that $\lambda_\pi = \frac{\varepsilon}{k}$.

this result is that from (26), we see that everything else the same, when ρ (and thereby, duration) decreases, then there will be a greater period by period incentive to increase $\hat{S}_t$, that is, keep nominal interest rates low to manage the debt burden. This, however, will increase current inflation further *in equilibrium* according to our result. As a consequence – and perhaps a little counterintuitively – equilibrium nominal interest rate will generally *increase* as well with lower duration since the real rate is exogenously given under flexible prices.[30] In particular, observe that $\hat{\imath}_t = E_t \pi_{t+1}$ (since $\hat{Y}_t = 0$ and there are no shocks) and hence i_t is increasing one-to-one with expected inflation.

b) Fully rigid prices Next we can consider the other extreme case: that of fully rigid prices. In this case, inflation is zero in equilibrium and hence $\pi_b = 0$. Then, one can directly consider the effects on the ex-ante real interest rate by analyzing the effect on the nominal interest rate since $r_b = i_b - \pi_b b_b = i_b$. Thus, we are interested in how i_b depends on the duration of debt. Using the parameters from Table 1, Fig (3) shows how i_b depends on duration of debt (quarters). We see that with shorter duration, there is more of an incentive to keep the nominal interest rate lower. The intuition for this result is again that from (26), when ρ (and thereby, duration) decreases, then there will be more of an incentive to increase $\hat{S}_t$, that is keep interest rates low, to manage the debt burden.

c) Partially rigid prices Having established the results in these two special cases, we now move on to the main mechanism in our paper in the intermediate and quantitatively relevant case of partially rigid prices. Fig (4) shows how r_b depends on duration of debt (quarters) at different levels of κ. When the duration of debt in the hands of the public is shorter, it unambiguously provides an incentive to the government to keep the short-term real interest rate lower in future. The intuition again is that by doing so, it reduces the cost of rolling over the debt period by period. That is, if the debt is short-term, then current real short rates will more directly affect the cost of rolling the debt over period by period, while the cost of rolling over long-term debt is not affected in the same way period by period. Also note that when prices are more flexible, that is when κ is bigger, r_b is affected less typically and as we saw above, in the extreme of fully flexible prices, (ex-ante) real interest rate is not controlled by policy at all.

Figs.(5) and (6) show how r_b depends on duration of debt at different levels of ε and λ_T respectively. We do this since these parameters affect the two weights in the loss-function and help us understand better the mechanics of the equilibrium. First, note that since $\lambda_\pi = \frac{\varepsilon}{k}$, changing ε will only affect the weight on inflation in the loss-function without affecting any private sector equilibrium conditions. The effect of a higher ε, while not very important quantitatively, is to increase the response of the real interest rate. This is because with a higher weight on inflation, since inflation responds less (we show this below), the real interest

[30] The way to understand how the model works in this case is to explore the government budget constraint. It is given by

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t$$

where again recall that $\hat{S}_t = -i_t + \rho\beta E_t \hat{S}_{t+1}$. Under flexible prices when there are no shocks (so $\hat{Y}_t = 0$) then $i_t = E_t \pi_{t+1} = \pi_b b_t$ and recall that $\hat{b}_t = b_b \hat{b}_{t-1}$. Solving for $\hat{S}_t$ and substituting for $\hat{\imath}_t$ we can then rewrite the budget constraint as

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + (1-\rho)\frac{1}{1-b_b\rho\beta}\pi_b\hat{b}_t - \psi\hat{T}_t.$$

The term $(1-\rho)\frac{1}{1-b_b\rho\beta}\pi_b\hat{b}_t$ reflects the cost of the government in rolling over the debt issued at t to t+1. Note that this cost depends on expected inflation which can be reduced by lowering current debt (because future inflation depends on this variable). The way current debt can be lowered is by creating inflation today (and thus depreciate nominal debt issued last period) or raising taxes. The government does both in equilibrium. The more long-term is the debt (the higher is ρ), however, the smaller is this effect since then a smaller part of the debt is rolled over on the current nominal interest rate (which depends directly on newly issued debt $\hat{b}_t$).

rate gets affected by a higher degree. Finally, the comparative statics with respect to λ_T are intuitive: as the weight on taxes in the loss-function increases, the real interest rate decreases by more in order to lower interest payments and reduce the need to raise taxes.

4.1.4 Inflation incentives and duration of outstanding debt

While the dependence of real interest rate incentive of the government on the duration of debt is the main mechanism of our paper, given the attention inflation incentives receive in the literature, we now study how π_b varies with the duration of outstanding government debt?[31] Moreover, it helps us emphasize a point that just focusing on inflation incentives might not be sufficient to understand the nature of optimal monetary and fiscal policy in a liquidity trap situation.

While an analytical expression for π_b is available in the appendix and it is possible to show that $\pi_b > 0$ for all $\rho < 1$, a full analytical characterization of the comparative statics with respect to the duration of debt is not available and so we rely on numerical results. Fig (7) below shows how π_b depends on duration of debt (quarters) at different levels of κ. As expected, π_b is positive at all durations in the figure.[32] More importantly, note that that at our baseline parameterization of $\kappa = 0.02$, π_b does decrease with duration over a wide range of maturity.[33] At the same time, however, there is a hump-shaped behavior, with π_b increasing when duration is increased at very short durations.

What drives this result? First, note from (26) that everything else the same, when ρ (and thereby, duration) decreases, then there will be an incentive to decrease $\hat{S}_t$, that is keep interest rates low to manage the debt burden. This will then increase inflation in equilibrium. At the same time however, the government's incentives on inflation are not fully/only captured by this reasoning. This is because what ultimately matters for the cost of debt is the real interest rate, which because of sticky prices, is endogenous and under the control of the central bank, and which we have seen before robustly depends negatively on the duration of debt. Therefore, to understand the overall effect on the government's inflation incentives, it is critical to analyze the targeting rule, as discussed above and given by (31). Here, the term $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]$ plays a key role as it captures the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$. Note first that β^{-1} here simply captures the role of surprise inflation in reducing the debt burden as government debt is nominal. This term would be present even when prices are completely flexible. The term $\kappa^{-1}(1-\rho)\sigma^{-1}$ however, appears because of sticky prices. This reflects how the real interest rate is affected by manipulation of the nominal interest rate and how it in turn affects output.

Thus, in situations where either κ^{-1}, σ^{-1}, or $(1-\rho)$ is high, this term can dominate and it can be the case that decreasing the duration (or decreasing ρ) actually leads to a lower π_b. This means, for example, that when κ^{-1} is low (or when prices are more flexible), the hump-shaped behavior of π_b gets restricted to very short maturities only as the channel coming from sticky-prices is not that influential. This is exactly what is seen in Fig (7): as κ increases, the range at which π_b increases with duration gets narrower and the hump disappears in fact for $\kappa = 3$.[34] This is also completely consistent with what we in fact showed above

[31] For some recent discussion and analysis of how inflation dynamics in sticky-price models could depend on duration of government debt, see Sims (2011) and Faraglia et al (2012).

[32] Again, it is possible to show analytically that for all $\rho < 1$, $\pi_b > 0$. In a very similar model but one with no steady-state debt, Eggertsson (2006) proved that $\pi_b > 0$ for $\rho = 0$ (that is, for one period debt).

[33] Some limited analytical results on the properties of π_b with respect to ρ are available in the appendix. For example, it can be shown that π_b is positive for all $\rho < 1$ and that when $\rho = 1 + \beta^{-1}\sigma\kappa$ (and thus, $\rho > 1$), $\pi_b = 0$. In this sense, for a specific case, one can show that π_b is declining in ρ by comparing some extreme cases (such as $\rho = 0$ with $\rho = 1 + \beta^{-1}\sigma\kappa$). Please see the appendix for details. Note also here that the upper bound on ρ is β^{-1}. So this case of $\pi_b = 0$ is not necessarily always reached.

[34] A similar picture is obtained when increasing σ. We do not present the figure to conserve space.

in the case of fully flexible prices, where inflation response depends negatively throughout on duration of debt.

Moreover, note that not surprisingly, π_b is higher at a given duration for higher κ. This simply reflects the fact that prices are now more flexible and inflation thus responds by more to a given level of outstanding debt. We also present some additional properties of π_b with respect to the two weights in the loss-function in order to understand the mechanics of the equilibrium. In Fig. (8) we see that as ε and thereby λ_π increases, as expected, inflation responds less to outstanding debt as inflation now has a greater weight in the loss-function. Fig. (9) presents results of changing the value of λ_T. As expected, now that taxes have a greater weight in the loss-function, inflation responds by more to outstanding debt in order to devalue debt and reduce the tax burden.

4.1.5 Debt dynamics and duration of outstanding debt

While the primary focus so far is on properties of r_b as it determines the real interest rate incentives of the central bank, it is also interesting to consider the properties of b_b, the parameter governing the persistence of government debt. This exercise is interesting in its own right, but more importantly, it is also worth exploring because as explained before, what is critical is the behavior of the real interest rate, and that gets affected by b_b as $E_t\pi_{t+1} = \pi_b\hat{b}_t = \pi_b b_b \hat{b}_{t-1}$. Unlike for π_b, it is not possible to show a tractable analytical solution (or any property) for b_b as it is generally a root of a fourth-order polynomial equation. We thus rely fully on numerical solutions.

Figs. (10)-(12) show how b_b depends on duration of debt at different levels of κ, ε, and λ_T respectively. It is clear that the persistence of debt increases monotonically as the duration increases.[35] In fact, for a high enough duration, debt dynamics approach that of a random walk ($b_b = 1$), as in the analysis of Barro (1979). The persistence of debt increases with duration mainly because the response of the short-term real interest rate decreases, as we discussed above. Some of this effect is reflected in the response of inflation decreasing as also discussed above. Thus, the existence of long-term nominal debt has an important impact on the dynamics of debt under optimal policy under discretion. Finally, intuitive comparative static results are in Fig. (12): as the loss-function weight on taxes increases, debt is less persistent at any level of debt duration.

Having established that at positive interest rates, decreasing the duration of debt increases the incentives of the government to lower short-term real interest rates, we now move on to analyzing the case where the nominal interest rate is at the zero lower bound.

4.2 At the zero lower bound

To model the case of a liquidity trap, we follow Eggertsson and Woodford (2003) and suppose that a large enough negative shock to the efficient rate of interest r_t^e, driven by an increase in the desire to save, makes the zero lower bound binding.[36] Moreover, we also assume that r_t^e follows a two-state Markov process with an absorbing state: every period, with probability μ, r_t^e takes a (large enough) negative value of r_L^e while with probability $1-\mu$, it goes back to steady-state and stays there forever after. This means that the

[35] There is thus, no hump-shaped pattern, unlike for inflation. The reason is that what matters directly for persistence of debt is the real interest rate and there is no hump-shaped pattern there as we show later. Note also that for one-period debt, often b_b is negative (even here, recall that π_b is still positive). A negative b_b is not very interesting empirically as it implies oscillatory behavior of debt.

[36] For an alternate way of generating a liquidity trap in monetary models, based on an exogenous drop in the borrowing limit, see Eggertsson and Krugman (2012).

economy will exit the liquidity trap with a constant probability of $1-\mu$ every period and that once it exits, it does not get into the trap again. The appendix contains details about the computation algorithm.

With this structure, we next consider the following policy experiment. At the liquidity trap, the level of debt is constant at b_L, while out of the trap, it is optimally determined by the government according to the Markov-perfect equilibrium described previously. Then, we analyze what would be the effect of changing the duration of debt once-and-for-all, while the zero lower bound is binding. In other words, we are interested in the comparative statics of the model as we vary the duration of debt at the liquidity trap. Note in particular that the steady-state market value of debt to taxes is always kept fixed in our experiments. For now a key abstraction is that the value of ρ is fixed, an issue we will come back to in the last section.

Our calibrated parameter values for this experiment are given in Table 1, where we pick values in a similar strategy to Eggertsson (2006). We pick μ and r_L^e to get a drop in output of 10%, to make the experiment relevant for the recent "Great Recession" in the United States, and a 2% percent drop in inflation. We allow for debt while at the liquidity trap to be 30% above its steady-state value, which as we mentioned before is in line with the Dallas Fed data. We will consider the experiment of reducing the duration of government debt by 7 months, starting from 16 quarters. For average maturity of government debt and the reduction in the duration of government debt due to quantitative easing by the Federal Reserve, we use the recent estimates of Chadha, Turner, and Zampoli (2013) which suggest that the average maturity of treasury debt held outside the Federal Reserve was around 4 years in the last 10 years and that recent Federal Reserve balance sheet policies reduced the maturity by around 7 months.

4.2.1 Initial duration of government debt

Figs. (13) and (14) show the response of inflation and output to a negative shock to the efficient rate of interest when the duration of government debt is 16 quarters. The solid line shows the expected path for the variables (impulse response) while the thin lines show each possible economic contingency (thus the third thin line – for example – corresponds to the case in which the shock reverts to normal in the third quarter). As is clear, because of the zero lower bound constraint, the economy suffers from deflation and because of the increase in the real interest rate that it creates (that is, a gap between the real interest rate and its efficient counterpart), also from a negative effect on output. This reflects what Eggertsson (2006) labelled the "deflation bias" of a discretionary central bank at the zero lower bound. In such a case, generating expectations of future inflation would be very beneficial as it would decrease the extent of deflation and the rise of the real interest rate, as stressed by Krugman (1998) and Eggertsson (2006). Observe, also, that because of the presence of nominal debt in the economy, the inflation rate once the shock is over is not zero. Instead it is of the order of about 1 percent. This "inflation bias" arises from the governments incentive to inflate away the nominal value of the outstanding debt.

It would be beneficial for the central bank to commit to keeping the short-term real interest rates lower in future, once the zero lower bound is no longer binding. This would decrease real long-term interest rates today and help spur the economy. Thus, one main goal of monetary policy would be to decrease the extent of increase in real interest rates (after the shock is over) that is seen in Fig. (15). In other words, as Krugman (1998) and Eggertsson (2006) emphasize, the central bank needs to "commit to being irresponsible."[37]

[37] Adam and Billi (2007), another important study of optimal policy in the New Keynesian model under discretion, emphasizes how the gains from commitment are much stronger once the zero lower bound on nominal interest rates is taken into account.

4.2.2 Shorter duration of government debt

A reduction in the duration of government debt outstanding helps achieve this goal. Figs. (16) and (17) show the response of inflation and output to a negative shock to the efficient rate of interest when the duration of government debt is 13.7 quarters. As is clear by comparing with the case when the duration is 16 quarters, at the trap, the extent of deflation is reduced by 95.4 basis points (annualized) as well as the negative effect on output by 140 basis points. Note in particular that once the shock is over and the zero lower bound is no longer a constraint, the response of inflation and output is higher compared to Figs. (13) and (14).

The main reason why this is achieved is that because the government's balance sheet has more long-term bonds (or its debt is more short-term), the central bank keeps the short-term real interest rates lower in future, especially once the zero lower bound is not binding, in order to keep the real interest rate low on the debt it is rolling over. Thus, quantitative easing indeed provides a signal about the future conduct of monetary policy and in particular, the future path of short-term interest rates. This then enables it to have effect on macroeconomic prices and quantities at the zero lower bound, as real interest rates also decline at the liquidity trap because of higher inflationary expectations once the trap is over. The response of the real interest rate when the duration of debt is shorter is given in Fig. (18), where by comparing with Fig. (15), one can see that the real interest rate increases by less at the trap and decreases by more out of the trap. This, then, is the central result of our paper: quantitative easing acts as a commitment device during a liquidity trap situation.[38]

We also conduct an experiment of how long the duration reduction has to be in order for the output gap to be closed completely at the zero lower bound. We find that approximately doubling the size of the quantitative easing, that is, reducing the maturity by 20.6 months, would make the response of output zero at the liquidity trap. Figs. (19)-(21) show the responses of inflation, output, and the real interest rate respectively in this case. As expected, this leads to a much bigger reduction in the real interest rate and much more of a boom in inflation and output out of the trap.

At this point, we want to clarify an important element of our analysis: the distinction between nominal and real interest rates. A key to understanding optimal policy in the sticky price model at the zero bound is that it involves committing to lower future short-term *real* interest rates. The short term real interest rate is the difference between future short-term nominal interest rate and expected inflation. A lower short-term real interest rate can be achieved by either a lower future short-term nominal interest rate or higher future expected inflation. A commitment of this kind will under some parameterization of the model be achieved via higher expected inflation and higher future short-term nominal interest rates – even if the difference between the two (the real rate) is going down. An implication of this is that a successful policy of quantitative easing aimed at lowering long-term real interest rate may involve an increase in expected future nominal interest rates and therefore, an increase in long-term nominal rates at the zero lower bound.

In fact, in our numerical experiments, this indeed does happen. Figs. (22) and (23) show the responses of the short and the long nominal interest rate when debt duration is 16 quarters while Figs. (24) and (25) show the responses when debt duration is 13.7 quarters.[39] As can be seen, the short-term nominal interest rate in fact rises by more once the trap is over when debt duration is 13.7 quarters and that the long-term nominal interest rate at the zero lower bound is higher by around 23 basis points. This aspect is relevant

[38] This result thus connects our paper with Persson, Persson, and Svensson (1987 and 2006), who show in a flexible price environment that a manipulation of the maturity structure of both nominal and indexed debt can generate an equivalence between discretion and commitment outcomes.

[39] Note that since we are plotting $\hat{\imath}_t$, the zero lower bound implies a bound of $-(1-\beta) = -0.01$.

because some empirical analyses of the effect of quantitative easing focus only on long-term nominal interest rates, which our analysis suggests is not a sufficient statistic. Even if long-term nominal interest rate decline very little in response to quantitative easing, or even if they increase (like in our example), this does not by itself suggest that the policy is ineffective, as long as the real interest rate is declining.[40]

4.2.3 Capital losses from reneging on optimal policy

We have emphasized and illustrated so far that the reason why lowering the duration of debt during a liquidity trap situation is beneficial is that it provides incentives for the government to keep the real interest rate low in future as it is now rolling over more short-term debt. We have shown these results by comparing the path of the real interest rate under optimal policy at a baseline and lower duration of debt.

Another way of framing this is that otherwise it would suffer capital losses on its balance sheet. These losses then would have to be accounted for by raising costly taxes. One way to illustrate the mechanism behind this result is to conduct the following thought experiment: suppose that once the liquidity trap is over, the government reneges on the path for inflation and output dictated by optimal policy under discretion and instead perfectly stabilizes them at zero. In such a situation, how large are capital losses, or equivalently, how high do taxes have to rise out of zero lower bound compared to if the government had continued to follow optimal policy? In particular, is this increase in taxes more when debt is of shorter duration ? We show in Figs. (26) and (27) the change in taxes if the government were to renege on optimal policy at a duration of 16 and 13.7 quarters respectively. As is clear, the increase in taxes out of zero lower bound are higher at a shorter duration of outstanding debt. Thus indeed, *lowering* the duration of government debt provides the government with *more* an incentive to keep the real interest rate low in future in order to avoid having to raise costly taxes.

4.2.4 Robustness

We now conduct some robustness exercises. In particular, compared to the literature, we calibrated some new parameters in this paper: λ_T (at 0.8) and b_L (at 0.30). We show in Fig. (28) the effect of quantitative easing on output when we vary λ_T and in Fig. (29) the effect of quantitative easing on output when we vary b_L.[41] Clearly, a higher λ_T increases the effect of quantitative easing as it leads to more of an incentive for the central bank to keep the real interest rate low (to avoid costly taxes) while a higher b_L also increases the effect of quantitative easing as it increases the debt burden and thereby, the incentive to keep real interest rates low. The Figures show that our results continue to hold qualitatively for several of these parameter values.

5 Extension

So far, we have focused on analyzing a situation where the duration of government debt is reduced once-and-for all. That is, we have studied comparative statics experiments with respect to ρ. A natural question that arises in this context is whether there is an incentive for the government to increase the duration of debt once the economy has recovered and if by not considering that, we are overstating our results.

[40] Admittedly, an increase in the long-term nominal interest rate because of quantitative easing is perhaps not empirically consistent. We just want to make the point that looking at the long-term nominal interest rate is not sufficient.

[41] Here we focus on the difference in output as a result of quantitative easing and to avoid clutter only show the probablity weighted impulse response function and not the various contingencies.

To address this, we now extend our model to allow the government to pick the duration of government debt optimally, period by period. That is, now, ρ is time-varying. In particular, the government issues a perpetuity bond in period t (B_t) which pays ρ_t^j dollars $j+1$ periods later. Following very similar manipulations as for the fixed duration case, the flow budget constraint of the government can be written as

$$S_t(\rho_t)b_t = (1 + \rho_{t-1}W_t(\rho_{t-1}))\, b_{t-1}\Pi_t^{-1} + (F - T_t) \tag{34}$$

where $S_t(\rho_t)$ is the period-t price of the government bond that pays ρ_t^j dollars $j+1$ periods later while $W_t(\rho_{t-1})$ is the period-t price of the government bond that pays ρ_{t-1}^j dollars $j+1$ periods later. Moreover, $b_t = \frac{B_t}{P_t}$. Given these types of government bonds, the asset-pricing conditions then take the form

$$S_t(\rho_t) = E_t\left[\beta \frac{u_C\left(C_{t+1}, \xi_{t+1}\right)}{u_C\left(C_t, \xi_t\right)} \Pi_{t+1}^{-1}\left(1 + \rho_t S_{t+1}(\rho_t)\right)\right] \tag{35}$$

$$W_t(\rho_{t-1}) = E_t\left[\beta \frac{u_C\left(C_{t+1}, \xi_{t+1}\right)}{u_C\left(C_t, \xi_t\right)} \Pi_{t+1}^{-1}\left(1 + \rho_{t-1} W_{t+1}(\rho_{t-1})\right)\right]. \tag{36}$$

The rest of the model is the same as before.

The government's instruments are now i_t, T_t , and ρ_t. Moreover, it is clear from above that in addition to b_{t-1}, now, ρ_{t-1} is also a state variable in the model. Then, we can write the discretionary government's problem recursively as

$$J\left(b_{t-1}, \rho_{t-1}, \xi_t\right) = \max\left[U\left(.\right) + \beta E_t J\left(b_t, \rho_t, \xi_{t+1}\right)\right]$$

subject to the three new constraints, (34)-(36), as well as the other private sector equilibrium conditions that are common from the model in the previous section. Here, $U\left(.\right)$ is the utility function of the household in (1) and $J(.)$ is the value function.[42] The detailed formulation of this maximization problem and the associated first-order necessary conditions are provided in the appendix. We discuss below why we take this non-linear as opposed to a linear-quadratic approach.

We proceed by computing the non-stochastic steady-state and then taking a first-order approximation of the non-linear government optimality conditions as well as the non-linear private sector equilibrium conditions around the steady-state. Of particular note is that a first-order approximation of (34)-(36) leads to

$$\hat{v}_t = \beta^{-1}\hat{v}_{t-1} - \beta^{-1}\hat{\pi}_t - (1-\rho)\,\hat{L}_t - \psi\hat{T}_t \tag{37}$$

where

$$\begin{aligned} \hat{v}_t &= \hat{b}_t + \frac{\rho\beta}{1-\rho\beta}\hat{\rho}_t \\ \hat{L}_t &= -\hat{\imath}_t + \rho\beta E_t \hat{L}_{t+1} \\ \hat{S}_t - \hat{W}_t &= \hat{\rho}_t - \hat{\rho}_{t-1}. \end{aligned} \tag{38}$$

Thus, after undertaking a transformation of variables as given by (38), (37) takes the same form as (26), the linearized government budget constraint when there was no time variation in duration. Thus, time-variation in duration does not play a separate role in government debt dynamics upto first order. Therefore, if we had taken a linear-quadratic approach, like before, then with the quadratic loss-function (28) and the linearized private sector equilibrium conditions including (37), we would not be able to solve for optimal debt duration

[42] Using compact notation, note that we can write the utility function as $u\left(C_t, \xi_t\right) + g\left(F - s(T_t - T), \xi_t\right) - \tilde{v}\left(Y_t, \xi_t\right).$

dynamics at all.

Given this appropriate redefinition of the state variable, we can show that the (bounded) solution of the model at positive interest rates takes the form

$$\begin{bmatrix} \hat{v}_t \\ \hat{\rho}_t \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ \rho_v & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_{t-1} \\ \hat{\rho}_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} \hat{Y}_t \\ \pi_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} Y_v & 0 \\ \pi_v & 0 \\ r_v & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_{t-1} \\ \hat{\rho}_{t-1} \end{bmatrix}$$

where, ρ_v, Y_v, π_v, and r_v are functions of the model parameters. Here, for simplicity, we are only focussing on solution of some of the endogenous model variables and do not consider shocks.

Two results stand out: first, $\hat{\rho}_t$ follows a random-walk like behavior; second, $\hat{\rho}_{t-1}$ does not affect directly other other variables such as output, inflation, and the real interest rate. This suggests then that even if we allow the government to pick the duration optimally period by period, there is no incentive for it to increase the duration of debt after the economy has recovered following a liquidity trap episode. Therefore, the simple comparative static analysis that we focussed on in the main part of the paper does not appear to be overstating our results.

6 Conclusion

We present a theoretical model where open market operations that reduce the duration of outstanding government debt, so called "quantitative easing," are not neutral because they affect the incentive structure of the central bank. In particular, in a Markov-perfect equilibrium of our model, reducing the duration of outstanding government provides an incentive for the central bank to keep short-term interest rates low in future in order to avoid balance sheet losses. When the economy is in a liquidity trap, such a policy is thus effective at generating inflationary expectations and lowering long-term interest rates, which in turn, helps mitigate the deflation and negative output gap that would ensue otherwise. In other words, quantitative easing is effective because it provides a "signal" to the private sector that the central bank will keep the short-term real interest rates low even when the zero lower bound is no longer a constraint in future.

In future work, it would be of interest to evaluate fully the quantitative importance of our model mechanism in a medium-scale sticky price model, along the lines of Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). Computation of Markov-perfect equilibrium under coordinated monetary and fiscal policy at the ZLB appears to not have been investigated for such models in the literature If one departs from the assumption of an efficient steady-state, which would preclude a linear-quadratic approach, this extension is likely to involve a substantial computation innovation. Moreover, as a methodological extension, it would be fruitful to allow for time-varying duration of government debt to affect real variables. To do so, it will be necessary to take a higher order approximation of the equilibrium conditions and modify the guess-and-verify algorithm to compute the Markov-perfect equilibrium accordingly.

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7 Tables and Figures

7.1 Tables

Table 1: Calibration of model parameters

Parameter	Value
β	0.99
σ	1
κ	0.02
ε	8
λ_T	0.8
$\frac{bS}{T}$	7.2

Table 2: Calibration of model parameters for the ZLB experiment

Parameter	Value
r_L^e	0.033
μ	0.75
b_L	0.30

7.2 Figures

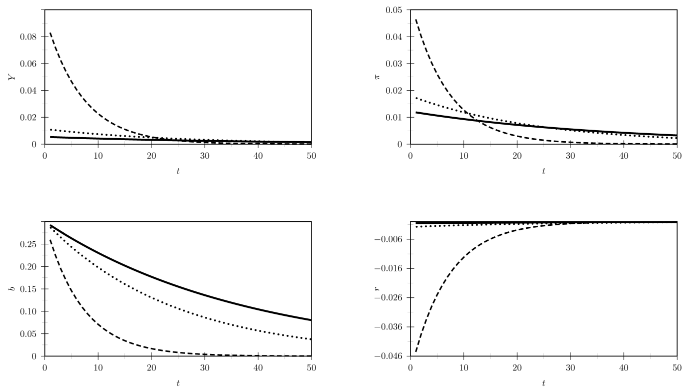

Figure 1: Responses at positive interest rates to a 30% increase in debt outstanding at 16, 13.7, and 8 qtrs

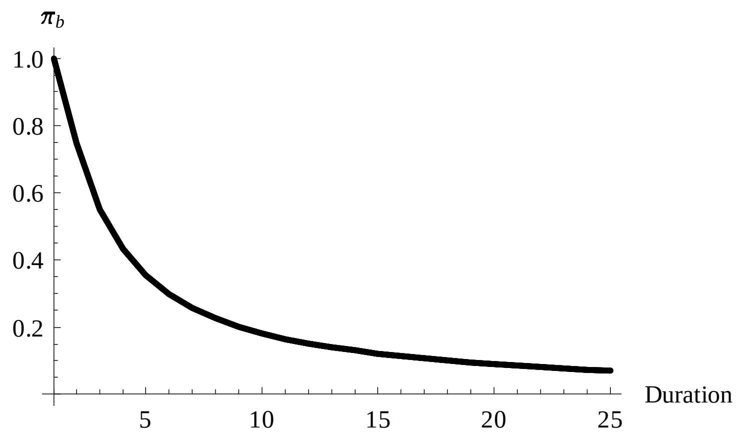

Figure 2: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt under fully flexible prices.

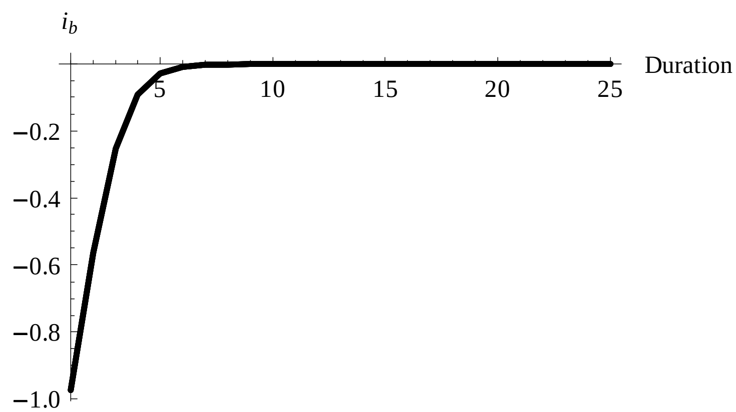

Figure 3: The effect on i_b of changing the duration (quarters) of outstanding nominal government debt under fully rigid prices.

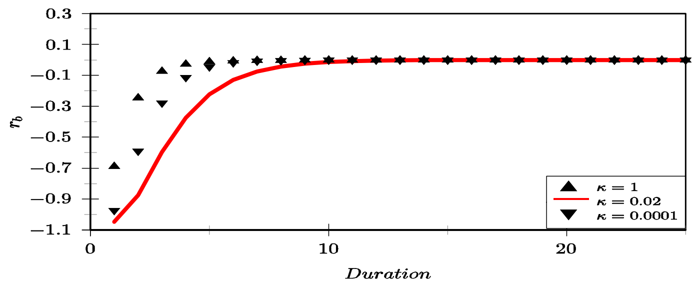

Figure 4: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

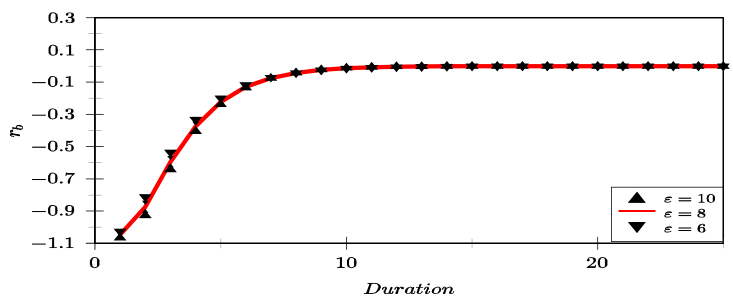

Figure 5: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

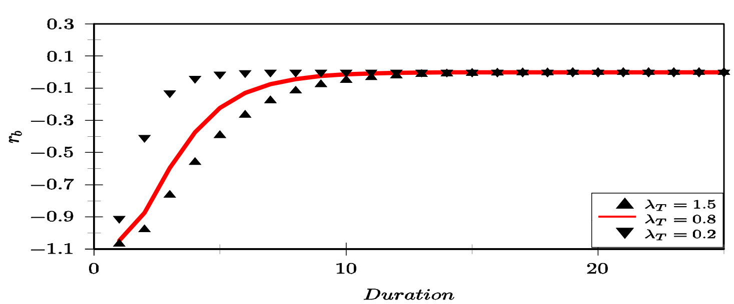

Figure 6: The effect on r_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

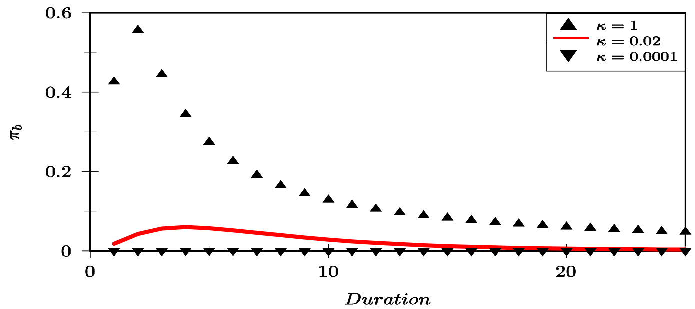

Figure 7: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

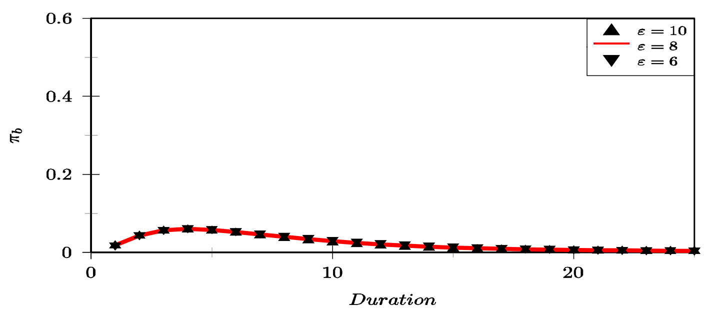

Figure 8: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

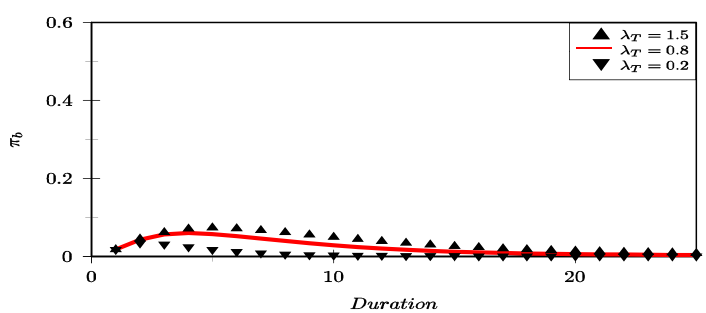

Figure 9: The effect on π_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

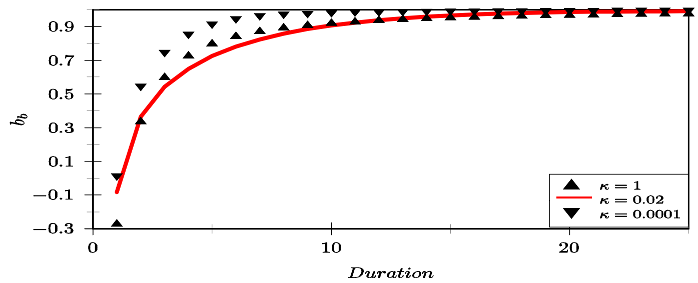

Figure 10: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the slope of the Phillips curve.

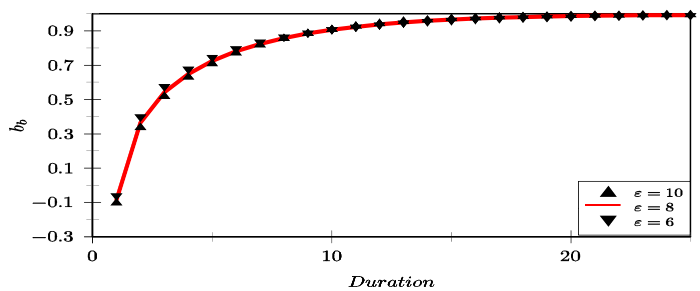

Figure 11: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the elasticity of substitution among varieties.

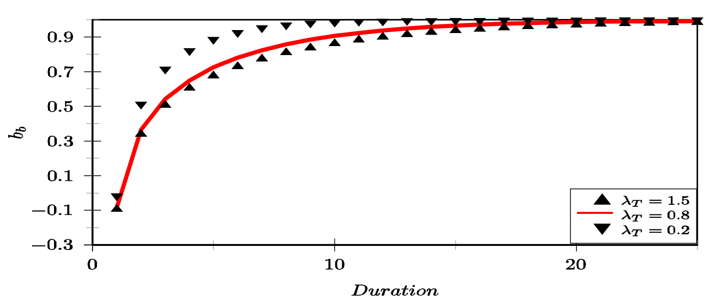

Figure 12: The effect on b_b of changing the duration (quarters) of outstanding nominal government debt at different levels of the weight on taxes in the loss function.

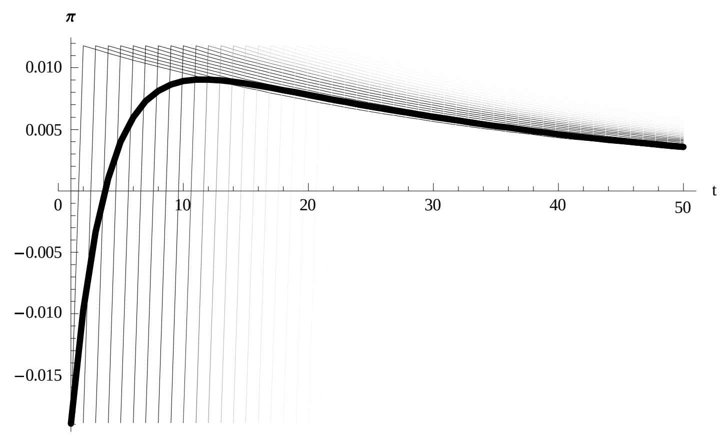

Figure 13: Reponse of inflation when the duration of government debt is 16 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

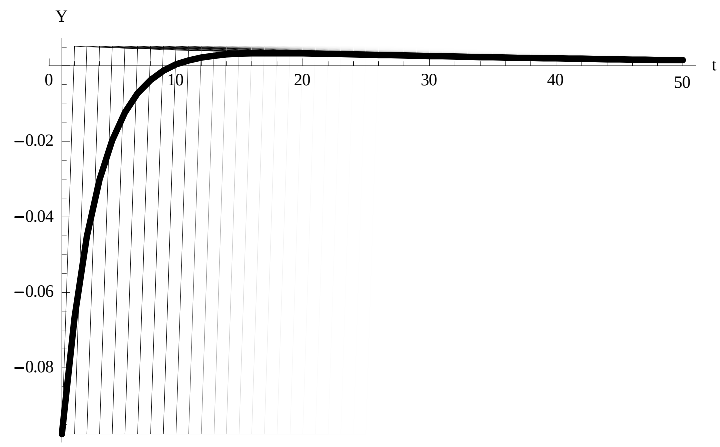

Figure 14: Reponse of output when the duration of government debt is 16 quarters. Each line represents the response of output when the efficient rate of interest returns to its steady-state in that period.

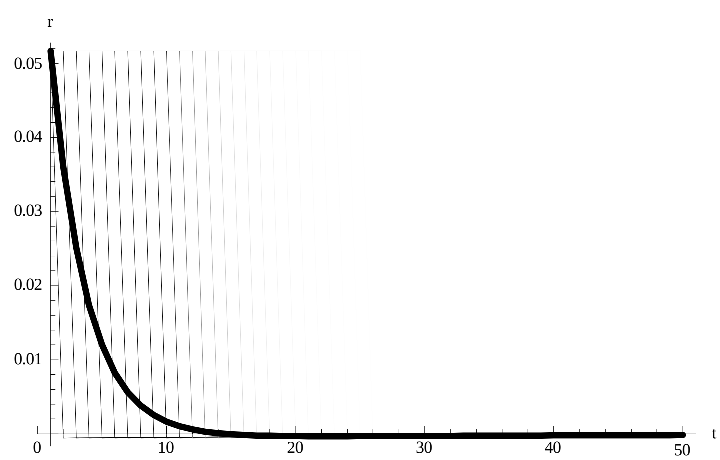

Figure 15: Reponse of the real interest rate when the duration of government debt is 16 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

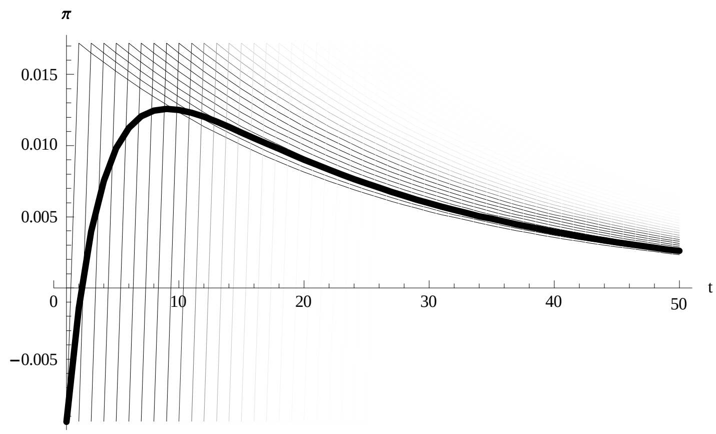

Figure 16: Reponse of inflation when the duration of government debt is 13.7 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

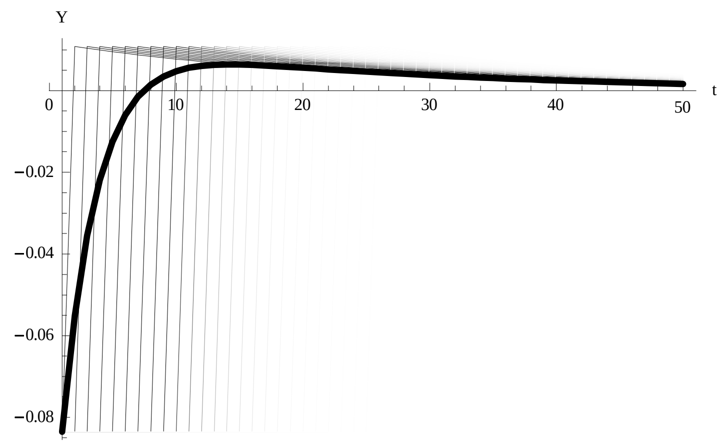

Figure 17: Reponse of output when the duration of government debt is 13.7 quarters. Each line represents the response of output when the efficient rate of interest returns to its steady-state in that period.

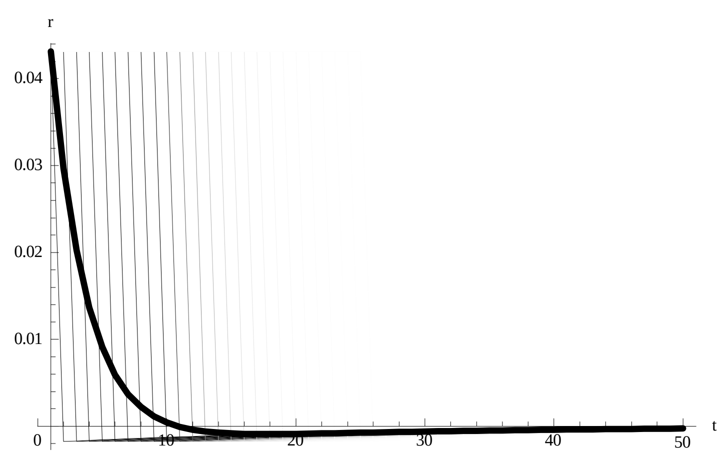

Figure 18: Reponse of the real interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

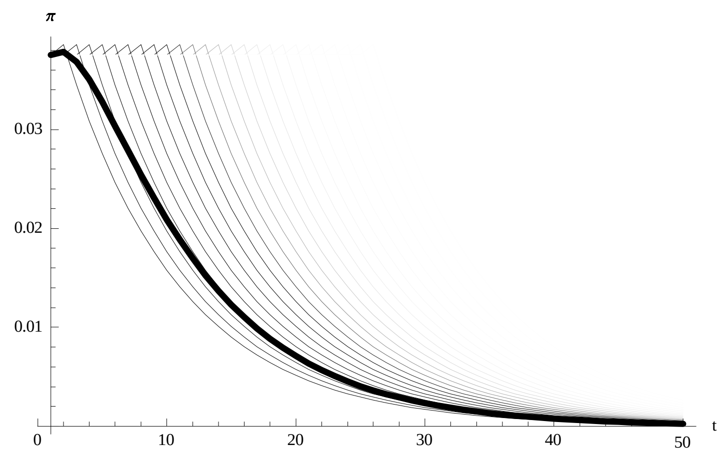

Figure 19: Reponse of inflation when the duration of government debt is 9.11 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

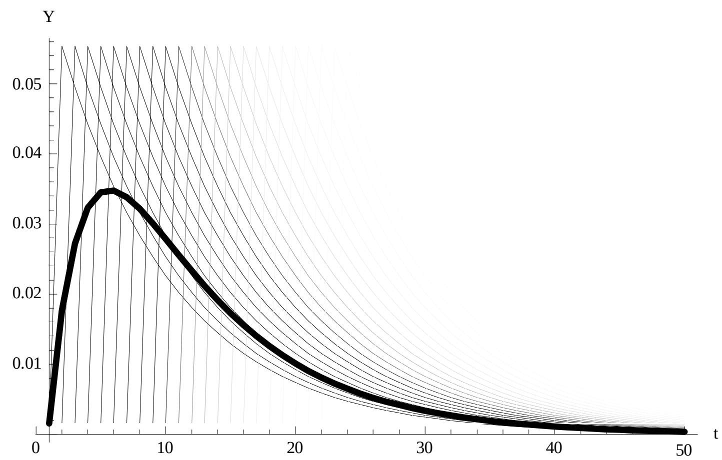

Figure 20: Reponse of output when the duration of government debt is 9.11 quarters. Each line represents the response of inflation when the efficient rate of interest returns to its steady-state in that period.

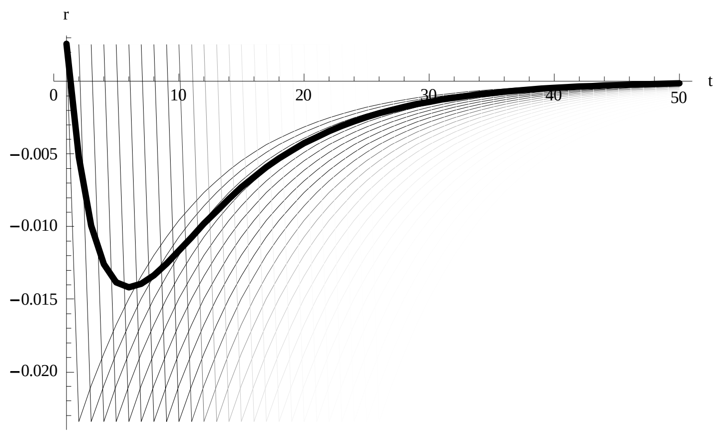

Figure 21: Reponse of the real interest rate when the duration of government debt is 9.11 quarters. Each line represents the response of the real interest rate when the efficient rate of interest returns to its steady-state in that period.

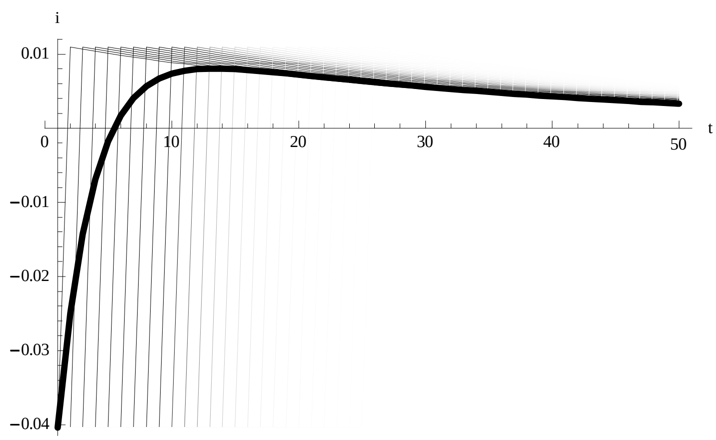

Figure 22: Reponse of the short-term nominal interest rate when the duration of government debt is 16 quarters. Each line represents the response of the short-term interest rate when the efficient rate of interest returns to its steady-state in that period.

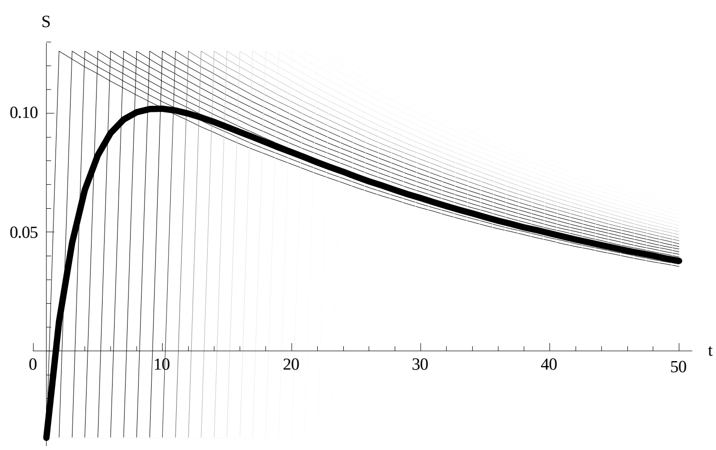

Figure 23: Reponse of the long-term nominal interest rate when the duration of government debt is 16 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

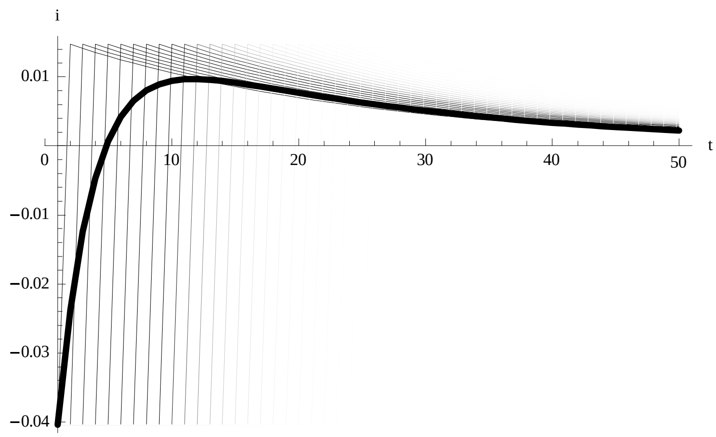

Figure 24: Reponse of the short-term nominal interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the short-term interest rate when the efficient rate of interest returns to its steady-state in that period.

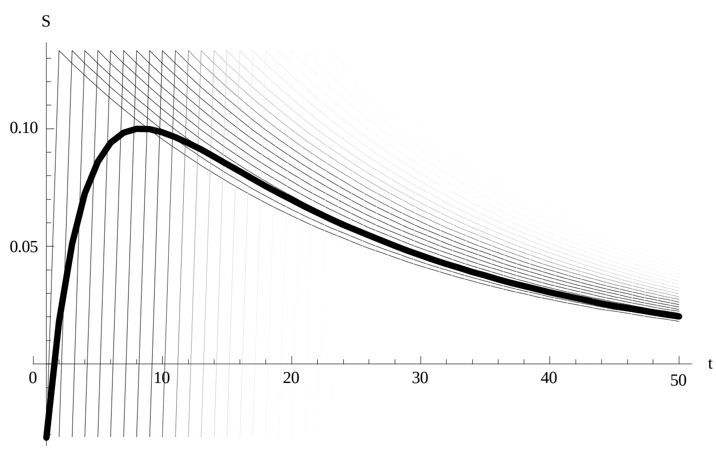

Figure 25: Reponse of the long-term nominal interest rate when the duration of government debt is 13.7 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

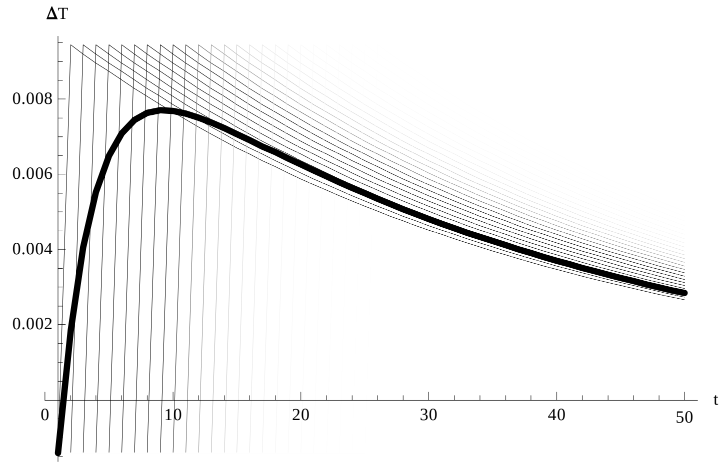

Figure 26: Increase in taxes a result of reneging on optimal policy when the duration of government debt is 16 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

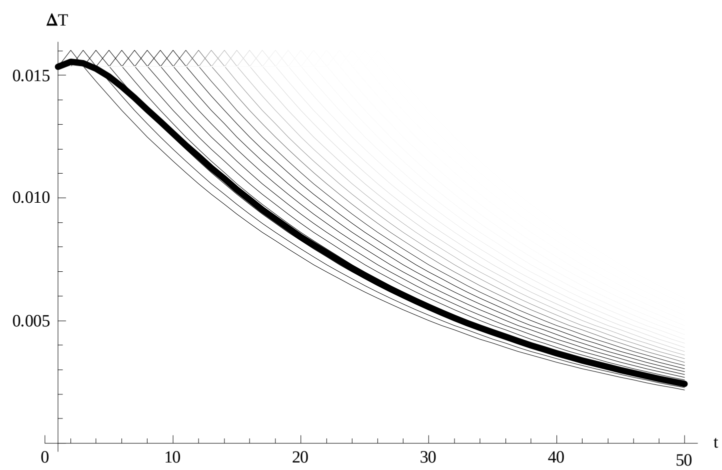

Figure 27: Increase in taxes a result of reneging on optimal policy when the duration of government debt is 13.7 quarters. Each line represents the response of the long-term interest rate when the efficient rate of interest returns to its steady-state in that period.

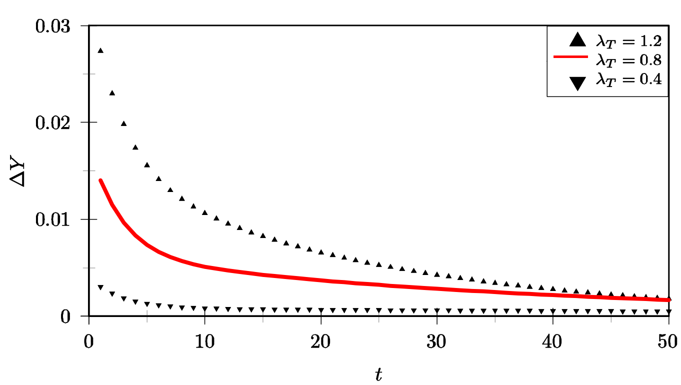

Figure 28: Effects of quantitative easing on output at different values of λ_T.

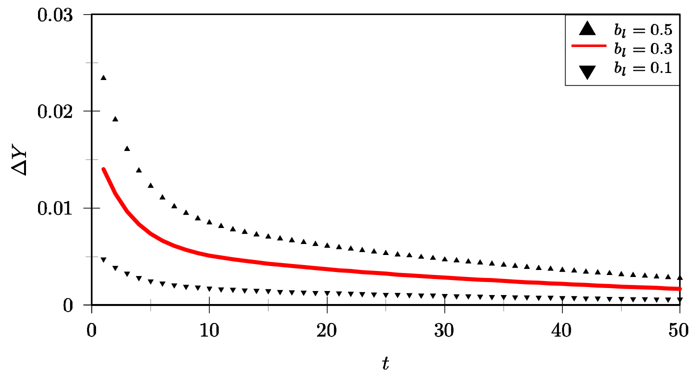

Figure 29: Effects of quantitative easing on output at different levels of initial debt.

8 Appendix

8.1 Model

8.1.1 The perpetuity nominal bond and the flow budget constraint

Following Woodford (2001), the perpetuity issued in period t pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$. The implied steady-state duration of this bond is then $(1-\beta\rho)^{-1}$.

Let the price of a newly issued bond in period t be $S_t(\rho)$. Given the existence of the unique stochastic discount factor $Q_{t,t+j}$, we can write this price as

$$S_t(\rho) = E_t \sum_{j=1}^{\infty} Q_{t,t+j}\rho^{j-1}.$$

Now consider the period $t+1$ price of such a bond that was issued in period t. We can then write the price $S^O_{t+1}(\rho)$

$$S^O_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-1}.$$

Note first that

$$S^O_{t+1}(\rho) = \rho S_{t+1}(\rho) \tag{39}$$

since

$$S_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-2}.$$

This is highly convenient since it implies that one needs to keep track, at each point in time, of the equilibrium price of only one type of bond.

Next, we derive an arbitrage condition between this perpetuity and a one-period bond. By simple expansion of the infinite sums above and manipulation of the terms, one gets

$$S_t(\rho) = [E_t Q_{t,t+1}] + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Since

$$E_t Q_{t,t+1} = \frac{1}{1+i_t}$$

we get

$$S_t(\rho) = \frac{1}{1+i_t} + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Substituting further for $S^O_{t+1}(\rho) = \rho S_{t+1}(\rho)$, we then derive

$$S_t(\rho) = \frac{1}{1+i_t} + \rho E_t\left[Q_{t,t+1}S_{t+1}(\rho)\right]. \tag{40}$$

Finally, consider the flow budget constraint of the government

$$B_t^S + S_t(\rho)B_t = (1+i_{t-1})\, B_{t-1}^S + \left(\rho^0 + S_t^O(\rho)\right) B_{t-1} + P_t\,(F_t - T_t)\,.$$

This can be simplified using $S_{t+1}^O(\rho) = \rho S_{t+1}(\rho)$ to

$$B_t^S + S_t(\rho)B_t = (1+i_{t-1})\, B_{t-1}^S + (1+\rho S_t(\rho))\, B_{t-1} + P_t\,(F_t - T_t)\,.$$

This is the form in which we write down the flow budget constraint of the household and the government in the main text.

8.1.2 Functional forms

We make the following functional form assumptions on preferences and technology

$$\begin{aligned}
u\,(C,\xi) &= \xi \bar{C}^{\frac{1}{\sigma}} \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \\
v\,(h(i),\xi) &= \xi\lambda \frac{h(i)^{1+\phi}}{1+\phi} \\
g\,(G,\xi) &= \xi \bar{G}^{\frac{1}{\sigma}} \frac{G^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \\
y(i) &= h(i)^{\kappa} \\
d(\Pi) &= d_1\,(\Pi-1)^2 \\
S\left(\bar{T}\right) &= s_1\,(T-T)^2
\end{aligned}$$

where we only consider a discount factor shock ξ. Note that $\xi = 1$ in steady-state and that in steady state, we scale hours such that $Y = 1$ as well. This implies that we can derive

$$\tilde{v}\,(Y,\xi) = \frac{1}{1+\phi} \lambda \xi Y^{\frac{1+\phi}{\kappa}}.$$

8.2 Efficient equilibrium

As benchmark, we first derive the efficient allocation.

Using $G_t = F_t - s(T_t - T) = F - s(T_t - T)$, the social planner's problem can be written as

$$\max u\,(C_t, \xi_t) + g\,(F - s(T_t - T)) - \tilde{v}\,(Y_t)$$

st

$$Y_t = C_t + F.$$

Formulate the Lagrangian

$$\begin{aligned}
L_t &= u\,(C_t, \xi_t) + g\,(F - s(T_t - T)) - \tilde{v}\,(Y_t) \\
&\quad + \phi_{1t}\,(Y_t - C_t - F)
\end{aligned}$$

FOCs (where all the derivatives are to be equated to zero)

$$\begin{aligned}
\frac{\partial L_t}{\partial Y_t} &= -\tilde{v}_Y + \phi_{1t} \\
\frac{\partial L_t}{\partial C_t} &= u_C + \phi_{1t}\,[-1] \\
\frac{\partial L_t}{\partial T_t} &= g_G\left(-s'(T_t - T)\right)
\end{aligned}$$

Eliminating the Lagrange multiplier gives

$$\begin{aligned}
u_C &= \tilde{v}_Y \\
g_G\left(-s'(T_t - T)\right) &= 0.
\end{aligned}$$

Note that we make the following functional form assumptions on the tax collection cost

$$s(0) = 0;\ s'(0) = 0.$$

Thus, when taxes are at steady state, that is, $T_t = T$, then $s(T_t - T) = s'(T_t - T) = 0$. But note that we will allow for $s''(0) > 0$.

Efficient allocation thus requires

$$u_C = \tilde{v}_Y$$
$$T_t = T.$$

In steady state, without aggregate shocks, we have

$$Y = C + F$$

$$u_C = \tilde{v}_Y$$
$$T_t = T.$$

8.3 Non-linear markov equilibrium

8.3.1 Optimal policy under discretion

The policy problem can be written as

$$J\left(b_{t-1}, \xi_t\right) = \max\left[U\left(\Lambda_t, \xi_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right)\right]$$

st

$$S_t(\rho) b_t = \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} + \left(F - T_t\right).$$
$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e}$$
$$i_t \geq 0$$
$$S_t(\rho) = \frac{1}{u_C\left(C_t, \xi_t\right)} \beta g_t^e$$
$$\varepsilon Y_t \left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t = \beta h_t^e$$
$$Y_t = C_t + F + d\left(\Pi_t\right)$$
$$f_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\right] = \bar{f}^e\left(b_t, \xi_t\right)$$
$$g_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\left(1 + \rho S_{t+1}(\rho)\right)\right] = \bar{g}^e\left(b_t, \xi_t\right)$$
$$h_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) d'\left(\Pi_{t+1}\right) \Pi_{t+1}\right] = \bar{h}^e\left(b_t, \xi_t\right)$$

Formulate the period Lagrangian

$$\begin{aligned}
L_t = {} & u\left(C_t, \xi_t\right) + g\left(F - s(T_t - T)\right) - \tilde{v}\left(Y_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right) \\
& + \phi_{1t}\left(S_t(\rho) b_t - \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} - \left(F - T_t\right)\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t, \xi_t\right)}{1 + i_t}\right) \\
& + \phi_{3t}\left(\beta g_t^e - u_C\left(C_t, \xi_t\right) S_t(\rho)\right) \\
& + \phi_{4t}\left(\beta h_t^e - \varepsilon Y_t\left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] - u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t\right) \\
& + \phi_{5t}\left(Y_t - C_t - F - d\left(\Pi_t\right)\right) \\
& + \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{2t}\left(g_t^e - \bar{g}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{3t}\left(h_t^e - \bar{h}^e\left(b_t, \xi_t\right)\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}$$

First-order conditions (where all the derivatives should be equated to zero)

$$\frac{\partial L_s}{\partial \Pi_t} = \phi_{1t}\left[(1+\rho S_t(\rho))\, b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{5t}\left[-d'\right]$$
$$\frac{\partial L_s}{\partial Y_t} = -\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}(1+s)\,u_C\right) + \varepsilon Y_t \tilde{v}_{yy} + \varepsilon \tilde{v}_y\right] + \phi_{5t}$$
$$\frac{\partial L_s}{\partial i_t} = \phi_{2t}\left[u_C(1+i_t)^{-2}\right] + \gamma_{1t}$$
$$\frac{\partial L_s}{\partial S_t} = \phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right]$$
$$\frac{\partial L_s}{\partial C_t} = u_C + \phi_{2t}\left[-u_{CC}(1+i_t)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t \frac{\varepsilon-1}{\varepsilon}(1+s)\,u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right]$$
$$\frac{\partial L_s}{\partial T_t} = g_G\left(-s'(T_t - T)\right) + \phi_{1t}$$
$$\frac{\partial L_s}{\partial b_t} = \beta E_t J_b\left(b_t, \xi_{t+1}\right) + \phi_{1t}\left[S_t(\rho)\right] + \psi_{1t}\left[-\bar{f}_b^e\right] + \psi_{2t}\left[-\bar{g}_b^e\right] + \psi_{3t}\left[-\bar{h}_b^e\right]$$
$$\frac{\partial L_s}{\partial f_t^e} = \beta\phi_{2t} + \psi_{1t}$$
$$\frac{\partial L_s}{\partial g_t^e} = \beta\phi_{3t} + \psi_{2t}$$
$$\frac{\partial L_s}{\partial h_t^e} = \beta\phi_{4t} + \psi_{3t}$$

The complementary slackness conditions are

$$\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t} i_t = 0$$

While the envelope condition is

$$J_b\left(b_{t-1}, \xi_t\right) = \phi_{1t}\left[-(1+\rho S_t(\rho))\,\Pi_t^{-1}\right]$$

This also implies that

$$\beta E_t J_b\left(b_t, \xi_{t+1}\right) = \beta E_t \phi_{1t+1}\left[-(1+\rho S_{t+1}(\rho))\,\Pi_{t+1}^{-1}\right]$$

8.3.2 Steady-state

A Markov-perfect steady-state is non-trivial to characterize because generally, we need to take derivatives of an unknown function, as is clear from the FOCs. Here, we will rely on the fact that given an appropriate production subsidy, the Markov-perfect steady-state will be the same as the efficient steady-state derived above.

First, note that this requires no resource loss from price-adjustment costs, which in turn requires

$$d(\Pi) = 0$$

and thereby ensures

$$Y = C + F$$

This means that we need

$$\Pi = 1.$$

Also, this implies

$$d'(\Pi) = 0.$$

Next, note from the Phillips curve that this means, we need

$$\frac{\varepsilon-1}{\varepsilon}(1+s)\,u_C - \tilde{v}_y = 0.$$

Now, since the efficient steady-state has $u_C = \tilde{v}_Y$, the production subsidy then has to satisfy

$$\frac{\varepsilon-1}{\varepsilon}(1+s) = 1.$$

We will be looking at a steady-state with positive interest rates

$$1 + i = \frac{1}{\beta}$$

which means that

$$\gamma_1 = 0$$

and that from the FOC wrt i_t we have

$$\phi_2 = 0.$$

Also, given that taxes are at steady-state, $g_G(-s'(T_t - T)) = 0$, from the FOC wrt T_t

$$\phi_1 = 0.$$

Given this, in turn, we have from the FOC wrt S_t

$$\phi_3 = 0.$$

Then, given that $d' = 0$ in steady-state and d'' is not, and since $\phi_1 = 0$, it gives from the FOC wrt to Π_t

$$\phi_4 = 0.$$

Note that this is highly convenient since these Lagrange multipliers being zero implies

$$\psi_1 = \psi_2 = \psi_3 = 0.$$

Thus, we do not need to worry about the derivatives of the unknown functions.

This proposed steady-state is consistent with other FOCs. For example, the FOC wrt Y_t is now given by

$$\phi_5 = \tilde{v}_Y$$

and that the FOC wrt C_t is given by

$$u_C = \phi_5$$

which implies

$$\tilde{v}_Y = u_C.$$

Finally, FOC wrt b_t implies

$$\beta J_b = \beta\phi_1\left[-(1 + \rho S(\rho))\,\Pi^{-1}\right] = 0$$

which is also consistent with the conjectured guess.

Finally, the guess of the steady-state is also consistent with the other model equilibrium conditions, with $S(\rho)$ given by

$$S(\rho) = \beta\left[(1 + \rho S(\rho))\right]$$

that is

$$S(\rho) = \frac{\beta}{1 - \rho\beta}.$$

Then b and F are linked by

$$S(\rho)b = (1 + \rho S(\rho))\,b + (F - T)$$

that is

$$T = F + \frac{1 - \beta}{1 - \rho\beta}b.$$

8.3.3 First-order approximation

We now take a log-linear approximation of the Markov perfect FOCs and the private sector equilibrium conditions around the steady-state above. Also, lets normalize the scale of the economy (with appropriate scaling of hours) so that $\bar{Y} = 1$. This implies $\bar{C} = 1 - F$. Also the shock ξ_t takes a value of 1 in steady-state.

Private sector equilibrium conditions We first start with the private sector equilibrium conditions. We denote variables that are in log-deviations from their respective steady-states by hats, except for $\hat{\imath}_t$.We denote variables in steady-state by bars.

First,

$$Y_t = C_t + F + d\,(\Pi_t)$$

gives

$$\hat{Y}_t = \bar{C}\hat{C}_t$$

Second,

$$\varepsilon Y_t \left[\frac{\varepsilon - 1}{\varepsilon}(1+s)\, u_C(C_t,\xi_t) - \tilde{v}_y(Y_t,\xi_t)\right] + u_C(C_t,\xi_t)\, d'(\Pi_t)\,\Pi_t = \beta E_t\left[u_C(C_{t+1},\xi_{t+1})\, d'(\Pi_{t+1})\,\Pi_{t+1}\right]$$

gives

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon\bar{u}_{CC}\bar{C}\hat{C}_t - \varepsilon\bar{v}_{yy}\hat{Y}_t - \varepsilon\bar{v}_{y\xi}\hat{\xi}_t + \varepsilon\bar{u}_{C\xi}\hat{\xi}_t = \beta\bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon(\bar{u}_{CC} - \bar{v}_{yy})\hat{Y}_t = \beta\bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\varepsilon(\bar{v}_{yy} - \bar{u}_{CC})}{\bar{u}_C d''}\hat{Y}_t.$$

Third,

$$1 + i_t = \frac{u_C(C_t,\xi_t)}{\beta E_t\left[u_C(C_{t+1},\xi_{t+1})\,\Pi_{t+1}^{-1}\right]}$$

gives

$$\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_{CC}\hat{Y}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

and

$$\hat{Y}_t = E_t\hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}}\left[\hat{\imath}_t - E_t\hat{\pi}_{t+1}\right] + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right]$$

Note here that this implies that the efficient rate of interest is given by

$$r_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right].$$

Fourth,

$$S_t(\rho) = \frac{1}{u_C(C_t,\xi_t)}\beta E_t\left[u_C(C_{t+1},\xi_{t+1})\,\Pi_{t+1}^{-1}(1+\rho S_{t+1}(\rho))\right]$$

gives

$$\bar{S}\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

which can be simplifies by making use of the log-linearized resource constraint above to yield

$$\bar{S}\bar{u}_{CC}\hat{Y}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}.$$

Note here that by using the log-linearized Euler equation above, one can further simplify as

$$\begin{aligned}
&\bar{S}\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\bar{S}\hat{S}_t \\
&= \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}
\end{aligned}$$

or

$$\begin{aligned}
&\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\hat{S}_t \\
&= \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{CC}E_t\hat{Y}_{t+1} + \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{u}_C E_t\hat{S}_{t+1}.
\end{aligned}$$

Moreover, since

$$\frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}} = 1$$

we have finally as the asset-pricing condition

$$\hat{\imath}_t + \hat{S}_t = \beta\rho E_t\hat{S}_{t+1}.$$

Fifth,

$$S_t(\rho)b_t = (1+\rho S_t(\rho))\, b_{t-1}\Pi_t^{-1} + (F - T_t)$$

gives

$$\hat{b}_t + \hat{S}_t = \rho\hat{S}_t + \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{b}_t - \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{\pi}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t$$

which is simplified further as

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\,\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

Then, finally, the expectation functions are given by

$$\hat{f}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C\hat{\pi}_{t+1}$$

$$\hat{g}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] = \left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

$$\hat{h}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] = \bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

Markov-perfect FOCs Here, note that since all the Lagrange multipliers except one are zero in steady-state, what we mean by hats will in fact only be deviations from steady-state for all the lagrange multipliers (as opposed to log-deviations).

First,

$$\phi_{1t}\left[\left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{5t}\left[-d'\right] = 0$$

gives

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{\phi}_5 d''\hat{\pi}_t = 0$$

and since $\bar{\phi}_5 = \tilde{v}_y = \bar{u}_C$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{u}_C d''\hat{\pi}_t = 0.$$

Second,

$$-\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t\tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{5t} = 0$$

gives

$$-\tilde{v}_{YY}\bar{Y}\hat{Y}_t + \left[\varepsilon\bar{Y}\tilde{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0.$$

Third,

$$\phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} = 0$$

gives

$$\bar{u}_C\beta^2\hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0.$$

Fourth,

$$\phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right] = 0$$

gives

$$\left[(1-\rho)\,\bar{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

Fifth,

$$u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right] = 0$$

gives

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

Sixth,

$$g_G\left(-s'(T_t - \bar{T})\right) + \phi_{1t} = 0$$

gives

$$-\bar{g}_G s^{''}\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

Seventh (after some replacements),

$$\beta E_t\phi_{1t+1}\left[-\left(1+\rho S_{t+1}(\rho)\right)\Pi_{t+1}^{-1}\right] + \phi_{1t}\left[S_t(\rho)\right] + \beta\phi_{2t}\left[\bar{f}_b^e\right] + \beta\phi_{3t}\left[\bar{g}_b^e\right] + \beta\phi_{4t}\left[\bar{h}_b^e\right] = 0$$

gives

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

8.4 Linear-quadratic approach

8.4.1 Linear approximation of equilibrium conditions

We approximate around an efficient non-stochastic steady-state where $\Pi = 1$. For simplicity, from here on we will assume that the only shock that hits the economy is a discount factor shock given by ψ. Standard manipulations that are prevalent in the literature, for example in Woodford (2003), and as shown above in the Markov perfect equilibrium, give (24) and (25) where $\sigma = \tilde{\sigma}\frac{C}{Y}$ and $k = \varepsilon \frac{(\sigma^{-1}+\phi)}{d''}$. Here we again detail the derivations of (26) and (27). Given

$$S_t(\rho) = \frac{1}{u_C(C_t,\xi_t)}\beta E_t\left[u_C(C_{t+1},\xi_{t+1})\,\Pi_{t+1}^{-1}(1+\rho S_{t+1}(\rho))\right],$$

and

$$1+i_t = \frac{u_C(C_t,\xi_t)}{\beta E_t\left[u_C(C_{t+1},\xi_{t+1})\,\Pi_{t+1}^{-1}\right]}$$

and the functional form assumptions above together with in steady state $1+i=\beta^{-1}$, log-linearization gives immediately

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}.$$

Next, given

$$S_t(\rho)b_t = (1+\rho S_t(\rho))\,b_{t-1}\Pi_t^{-1} + (F_t - T_t)$$

and that we assume $F_t = F$ and have from steady state $S = \frac{\beta}{1-\rho\beta}$, $T = F + \frac{1-\beta}{1-\rho\beta}b$, log-linearization gives immediately

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t.$$

Note that also the following relationship holds in steady state $F = G$. We finally derive an expression for r_t^e, the efficient rate of interest

$$r_t^e = \tilde{\sigma}^{-1}(\psi_t - E_t\psi_{t+1}).$$

8.4.2 Quadratic approximation of household utility

For household utility, we need to approximate the following three components

$$u(Y_t - F - d(\Pi_t),\xi_t);\ g(F - S(T_t - T),\xi_t);\ v(Y_t,\xi_t).$$

Standard manipulations that are prevalent in the literature, for example in Woodford (2003), give as a second-order approximation to household utility

$$\frac{1}{2}\frac{-\sigma\left(\hat{T}_t^2\left(F-\bar{Y}\right)s''\left(\bar{T}\right) + Fd''(1)\hat{\Pi}_t^2\right) + 2\sigma\hat{Y}_t\left(-\bar{Y} + F\hat{\xi}_t + F\right) + \hat{Y}_t^2}{\sigma\left(F-\bar{Y}\right)} +$$

$$\frac{1}{2}\bar{Y}\left(\frac{d''(1)\hat{\Pi}_t^2}{F-\bar{Y}} + \hat{\xi}_t\left(\frac{2\sigma}{\sigma-1} - \frac{2\hat{Y}_t}{F-\bar{Y}}\right) + \frac{2\sigma}{\sigma-1}\right) - \frac{1}{2}2\lambda\left(\hat{\xi}_t+1\right)\hat{Y}_t\bar{Y}^{\phi} - \frac{1}{2}\frac{2\lambda\left(\hat{\xi}_t+1\right)\bar{Y}^{\phi+1}}{\phi+1} - \frac{1}{2}\lambda\phi\hat{Y}_t^2\bar{Y}^{\phi-1} + tip$$

which is in turn given as

$$\frac{1}{2}\left(\hat{Y}_t^2\left(\frac{1}{F\sigma - \sigma\bar{Y}} - \lambda\phi\bar{Y}^{\phi-1}\right) + \hat{T}_t^2\left(-s''\left(\bar{T}\right)\right) - 2\left(\hat{\xi}_t+1\right)\hat{Y}_t\left(\lambda\bar{Y}^{\phi}-1\right) - d''(1)\hat{\Pi}_t^2\right) + tip.$$

Now lets multiply everything by $\frac{1}{\phi+\tilde{\sigma}^{-1}}$ and consider efficient equilibrium in steady-state ($u_C = \tilde{v}_Y$), together with the scaling that $\lambda\bar{Y}^{\phi} = 1$ and $\bar{Y} = \bar{C} + F = 1$, to get

$$-\frac{\tilde{\sigma}\hat{T}_t^2 s''\left(\bar{T}\right)}{2\left(\phi\tilde{\sigma}+1\right)} - \frac{\tilde{\sigma}d''(1)\hat{\Pi}_t^2}{2\left(\phi\tilde{\sigma}+1\right)} - \frac{\hat{Y}_t^2}{2}$$

So, finally, we get as approximation

$$-\left[\lambda_\pi\pi_t^2 + \hat{Y}_t^2 + \lambda_T\hat{T}_t^2\right]$$

where

$$\lambda_T = \frac{s''\left(\bar{T}\right)}{\phi+\tilde{\sigma}^{-1}}$$

$$\lambda_\pi = \frac{d''(1)}{(\phi + \tilde{\sigma}^{-1})} = \frac{\varepsilon}{\kappa}.$$

8.4.3 Markov-perfect equilibrium at positive interest rates

Given the Lagrangian where the expectation functions are substituted and the shocks are suppressed

$$L_t = \frac{1}{2}(\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2) + \beta E_t V(b_t, r_{t+1}^e) + \phi_{1t}[\hat{Y}_t - Y_b b_t + \sigma \hat{\imath}_t - \sigma \pi_b b_t - \sigma r_t^e]$$
$$+ \phi_{2t}[\pi_t - \kappa \hat{Y}_t - \beta \pi_b b_t] + \phi_{3t}[b_t - \beta^{-1} b_{t-1} + \beta^{-1}\pi_t + (1-\rho)\hat{S}_t + \psi \hat{T}_t] + \phi_{4t}[\hat{S}_t + \hat{\imath}_t - \rho\beta S_b b_t]$$

the first-order necessary conditions are given by

$$\frac{\partial L}{\partial \pi_t} = \lambda_\pi \pi_t + \phi_{2t} + \beta^{-1}\phi_{3t} = 0$$
$$\frac{\partial L}{\partial Y_t} = Y_t + \phi_{1t} - \kappa \phi_{2t} = 0$$
$$\frac{\partial L}{\partial T_t} = \lambda_T T_t + \psi \phi_{3t} = 0$$
$$\frac{\partial L}{\partial i_t} = \sigma \phi_{1t} + \phi_{4t} = 0$$
$$\frac{\partial L}{\partial S_t} = \phi_{3t}(1-\rho) + \phi_{4t} = 0$$
$$\frac{\partial L}{\partial b_t} = \beta E_t V_b(b_t, r_{t+1}^e) - (Y_b + \sigma \pi_b)\phi_{1t} - \beta \pi_b \phi_{2t} + \phi_{3t} - \rho \beta S_b \phi_{4t} = 0$$

while the envelope condition is given by

$$V_b(b_{t-1}, r_t^e) = -\beta^{-1}\phi_{3t}$$

which implies

$$E_t V_b(b_t, r_{t+1}^e) = -\beta^{-1} E_t \phi_{3t+1}.$$

We can then combine the envelope condition with the last FOC to yield

$$-E_t \phi_{3t+1} - (Y_b + \sigma \pi_b)\phi_{1t} - \beta \pi_b \phi_{2t} + \phi_{3t} - \rho\beta S_b \phi_{4t} = 0.$$

To summarize, we have

$$\lambda_\pi \pi_t + \phi_{2t} + \beta^{-1}\phi_{3t} = 0$$

$$Y_t + \phi_{1t} - \kappa \phi_{2t} = 0$$
$$\lambda_T T_t + \psi \phi_{3t} = 0$$
$$\sigma \phi_{1t} + \phi_{4.t} = 0$$
$$\phi_{3t}(1-\rho) + \phi_{4t} = 0$$
$$-E_t \phi_{3t+1} - (Y_b + \sigma \pi_b)\phi_{1t} - \beta \pi_b \phi_{2t} + \phi_{3t} - \rho\beta S_b \phi_{4t} = 0$$
$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t \pi_{t+1} - r_t^e)$$
$$\pi_t = \kappa \hat{Y}_t + \beta E_t \pi_{t+1}$$
$$b_t = \beta^{-1} b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi \hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t \hat{S}_{t+1}$$

which can be simplified to get

$$\lambda_\pi \pi_t + \kappa^{-1} Y_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T \hat{T}_t$$
$$[1 - \beta\pi_b \kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)]\hat{T}_t = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}Y_t + E_t\hat{T}_{t+1}$$
$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t \pi_{t+1} - r_t^e)$$
$$\pi_t = \kappa \hat{Y}_t + \beta E_t \pi_{t+1}$$
$$b_t = \beta^{-1} b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi \hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t \hat{S}_{t+1}.$$

The final step is then to match coefficients after replacing the conjectured solutions

$$\lambda_\pi \pi_b b_{t-1} + \kappa^{-1}(Y_b b_{t-1}) = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T (T_b b_{t-1})$$

$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)][T_b b_{t-1}] = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}[Y_b b_{t-1}] + [T_b(b_b b_{t-1})]$$

$$Y_b b_{t-1} = Y_b[b_b b_{t-1}] - \sigma((i_b b_{t-1}) - [\pi_b[b_b b_{t-1}]])$$

$$\pi_b b_{t-1} = \kappa[Y_b b_{t-1}] + \beta\pi_b[b_b b_{t-1}]$$

$$b_b b_{t-1} = \beta^{-1} b_{t-1} - \beta^{-1}[\pi_b b_{t-1}] - (1-\rho)[S_b b_{t-1}] - \psi[T_b b_{t-1}]$$

$$S_b b_{t-1} = -(i_b b_{t-1}) + \rho\beta S_b[b_b b_{t-1}]$$

which in turn can be simplified to get

$$\lambda_\pi\pi_b + \kappa^{-1}Y_b = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T T_b \tag{41}$$

$$[(1-\rho)^{-1} - \beta\pi_b\kappa^{-1}\sigma^{-1} - (Y_b + \sigma\pi_b)\sigma^{-1} + \rho\beta S_b](1-\rho)T_b = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}Y_b + T_b b_b \tag{42}$$

$$Y_b = Y_b b_b - \sigma(i_b - \pi_b b_b) \tag{43}$$

$$\pi_b = \kappa Y_b + \beta\pi_b b_b \tag{44}$$

$$b_b = \beta^{-1} - \beta^{-1}\pi_b - (1-\rho)S_b - \psi T_b \tag{45}$$

$$S_b = -i_b + \rho\beta S_b b_b. \tag{46}$$

We now show some properties of π_b analytically. First, note that we will be restricting to stationary solutions, that is one where $| b_b | < 1$. Manipulations of (41)-(46) above lead to the following closed-form expression for π_b

$$\pi_b = \frac{(1-\beta b_b)}{\beta\left[\frac{T}{lV}\chi[\lambda_\pi + \kappa^{-1}\kappa^{-1}(1-\beta b_b)] + [(1-\rho)\sigma^{-1}\kappa^{-1}(1-b_b) + \beta^{-1}](1-\beta b_b)(1-\rho\beta b_b)^{-1}\right]}$$

$$\text{where } \chi = \left[[\kappa^{-1}\sigma^{-1}(1-\rho) + \beta^{-1}]\frac{bS}{T}\lambda_T\right]^{-1}.$$

Since $| b_b | < 1$, it is clear that $\pi_b > 0$ for all $\rho < 1$,. What happens when$\rho > 1$? Numerically, we have found that often $\pi_b > 0$ still, but sometimes in fact it can hit zero (and then actually go negative if ρ is increased further). It is in fact possible to pin-down analytically when $\pi_b = 0$. Note that from (44), if $\pi_b = 0$, then $Y_b = 0$. This then implies that for this to be supported as a solution for all possible values of T_b, from (41), it must be the case that

$$[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}] = 0$$

which in turn implies that

$$\rho = 1 + \beta^{-1}\sigma\kappa.$$

In this knife-edge case, note that one needs $\rho > 1$. Moreover, since the upper bound on ρ is β^{-1}, one needs to ensure that

$$1 + \beta^{-1}\sigma\kappa < \beta^{-1} \text{ or } \sigma\kappa < 1 - \beta.$$

This is a fairly restrictive parameterization (not fulfilled in our baseline, for example). Still, it is instructive to note that in this case of when π_b does reach 0, then it is indeed possible to show analytically that π_b declines as duration is increased while comparing $\rho = 0$ with $\rho = 1 + \beta^{-1}\sigma\kappa$ or $\rho = 1$ with $\rho = 1 + \beta^{-1}\sigma\kappa$.

8.5 Equivalence of the two approaches

We now show the equivalence of the linearized dynamic systems for non-linear and linear-quadratic approaches.

Consider the system of equations describing private sector equilibrium and optimal government policy

$$\hat{Y}_t = E_t\hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}}[\hat{\imath}_t - E_t\hat{\pi}_{t+1} - r_t^e]$$

$$\hat{\pi}_t = \frac{\varepsilon(\bar{v}_{yy} - \bar{u}_{CC})}{\bar{u}_C d''}\hat{Y}_t + \beta E_t\hat{\pi}_{t+1}.$$

$$\hat{\imath}_t + \hat{S}_t = \beta\rho E_t\hat{S}_{t+1}.$$

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\,\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{\phi}_5 d''\hat{\pi}_t = 0$$

$$-\bar{v}_{YY}\overline{Y}\hat{Y}_t + \left[\varepsilon\bar{Y}\bar{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0$$

$$\bar{u}_C\beta^2\hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0$$

$$\left[(1-\rho)\,\bar{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

$$-\bar{g}_G s^{''}\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

Under additional functional assumptions outlined in previous sections we get

$$\bar{v}_{YY} = \phi\overline{Y}^{-1} = \phi$$

$$\bar{u}_{CC} = -\sigma^{-1}\bar{C}^{-1} = -\tilde{\sigma}^{-1}$$

$$\bar{u}_{C\xi} = 1$$
$$\bar{u}_C = 1$$

$$\bar{g}_G = 1$$

$$\bar{\phi}_5 = 1$$

$$\bar{Y} = 1$$

The first four equations are equivalent to their counterparts in the LQ-approach once one use the functional form assumptions to get

$$\frac{\bar{u}_C}{\bar{u}_{CC}} = -\tilde{\sigma}$$

and introduce new notation

$$\frac{\varepsilon\left(\bar{v}_{yy} - \bar{u}_{CC}\right)}{\bar{u}_C d''} = \kappa.$$

The latter relation implies

$$\frac{\varepsilon\left(\phi + \tilde{\sigma}^{-1}\right)}{\kappa} = d''.$$

Let us guess solutions for all variables for the case when the ZLB is slack as a linear function of $\hat{b}_{t-1}$ and $\hat{r}^e_t$. Then the expectations will take form

$$\hat{f}^e_t = \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1} = \hat{f}^e_b\bar{b}\hat{b}_{t-1} + \hat{f}^e_r\ \hat{r}^e_t$$
$$\hat{g}^e_t = \left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1} = \hat{g}^e_b\bar{b}\hat{b}_{t-1} + \hat{g}^e_r\ \hat{r}^e_t$$
$$\hat{h}^e_t = \bar{u}_C d'' E_t\hat{\pi}_{t+1} = \hat{h}^e_b\bar{b}\hat{b}_{t-1} + \hat{h}^e_r\ \hat{r}^e_t$$

where

$$\hat{f}^e_b = -\bar{b}^{-1}\left(\tilde{\sigma}^{-1}Y_b + \pi_b\right)$$
$$\hat{g}^e_b = -\tilde{\sigma}^{-1}\bar{b}^{-1}\left(1+\rho\bar{S}\right)Y_b - \left(1+\rho\bar{S}\right)\bar{b}^{-1}\pi_b + \rho\bar{S}S_b\bar{b}^{-1}$$

$$\hat{h}_b^e = \frac{\varepsilon\left(\phi + \tilde{\sigma}^{-1}\right)}{\kappa}\pi_b\bar{b}^{-1}$$

Also under the assumption about the process $\hat{\xi}_t$ we get

$$E_t\hat{\xi}_{t+1} = \mu\hat{\xi}_t$$

and

$$\hat{r}_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C}\left[\mu\hat{\xi}_t - \hat{\xi}_t\right] = \hat{\xi}_t(1-\mu).$$

When the economy is out of ZLB $\hat{r}_t^e = 0$.

Now we can find explicit representation for the lagrange multipliers

$$\hat{\phi}_{1t} = s^{''}\bar{T}\hat{T}_t.$$

$$\hat{\phi}_{2t} = 0$$

$$\hat{\phi}_{3t} = [(1-\rho)]\, s^{''}\bar{T}\hat{T}_t.$$

$$\hat{\phi}_{4t} = \frac{\kappa}{\varepsilon\left(\phi + \tilde{\sigma}^{-1}\right)}\left(1 + \rho\bar{S}\right)\bar{b}s^{''}\bar{T}\hat{T}_t - \hat{\pi}_t$$

$$\hat{\phi}_{5t} = \phi\hat{Y}_t - \phi\frac{\kappa}{\left(\phi + \tilde{\sigma}^{-1}\right)}\left(1 + \rho\bar{S}\right)s^{''}\bar{T}\hat{T}_t + \varepsilon\phi\hat{\pi}_t$$

So the last two equations take the form

$$\varepsilon\hat{\pi}_t = -\hat{Y}_t + \frac{1}{\left(\phi + \tilde{\sigma}^{-1}\right)}\left[\tilde{\sigma}^{-1}\bar{S}\left(1-\rho\right) + \kappa\left(1 + \rho\bar{S}\right)\right]\bar{b}s^{''}\bar{T}\hat{T}_t.$$

$$\bar{S}s^{''}\bar{T}\hat{T}_t + \left(\rho\beta S_b - \tilde{\sigma}^{-1}Y_b - \pi_b\right)\bar{b}^{-1}\left(1-\rho\right)\bar{S}\bar{b}s^{''}\bar{T}\hat{T}_t + \bar{b}^{-1}\pi_b\bar{S}\bar{b}s^{''}\bar{T}\hat{T}_t - \beta\frac{\varepsilon\left(\phi + \tilde{\sigma}^{-1}\right)}{\kappa}\pi_b\bar{b}^{-1}\hat{\pi}_t = s^{''}\bar{T}\bar{S}E_t\hat{T}_{t+1}.$$

which after straightforward manipulations become

$$\frac{\varepsilon}{\kappa}\hat{\pi}_t + \hat{Y}_t\kappa^{-1} = \left[\tilde{\sigma}^{-1}\left(1-\rho\right)\kappa^{-1} + \beta^{-1}\right]\frac{\bar{b}\bar{S}}{\bar{T}}\frac{\bar{T}^2 s^{''}}{\left(\phi + \tilde{\sigma}^{-1}\right)}\hat{T}_t.$$

$$\left(1 + \left(\rho\beta S_b - \tilde{\sigma}^{-1}Y_b - \pi_b\right)\left(1-\rho\right) + \pi_b\right)\frac{\bar{S}\bar{b}}{\bar{T}}\frac{s^{''}\bar{T}^2}{\left(\phi + \tilde{\sigma}^{-1}\right)}\hat{T}_t - \beta\frac{\varepsilon}{\kappa}\pi_b\hat{\pi}_t = \frac{s^{''}\bar{T}^2}{\left(\phi + \tilde{\sigma}^{-1}\right)}\frac{\bar{b}\bar{S}}{\bar{T}}E_t\hat{T}_{t+1}.$$

These two equations are equivalent for the last two equations from the dynamic system obtained in LQ–approach once we use the derived weights from the quadratic approximation of the loss function

$$\lambda_T = \frac{s^{''}\bar{T}^2}{\left(\phi + \tilde{\sigma}^{-1}\right)}$$

$$\lambda_\pi = \kappa^{-1}\varepsilon.$$

8.6 Computation at ZLB

In our experiment the debt is kept fixed at the zero lower bound at b_L . Moreover, at the zero lower bound, $\hat{\imath}_t = 1 - \beta^{-1}$. Given the specific assumptions on the two-state Markov shock process, the equilibrium is described by the system of equations

$$\begin{aligned}
\hat{Y}_t &= -\sigma\left(-\pi_b(1-\mu)\hat{b}_t - r_t^e - \mu\pi_L + \hat{\imath}_t\right) + (1-\mu)\hat{b}_t Y_b + \mu Y_L,\\
\hat{\pi}_t &= \beta\left(\pi_b(1-\mu)\hat{b}_t + \mu\pi_L\right) + \kappa Y_t,\\
\hat{\imath}_t &= 1 - \beta^{-1},\\
\hat{b}_t &= b_L,\\
\hat{S}_t &= -\hat{\imath}_t + \rho\beta\left((1-\mu)S_b b_L + \mu S_L\right).
\end{aligned}$$

where variables with a b subscript denote the solution we compute at positive interest rates while variables with a L subscript denote values at the ZLB. The solution to this system is

$$\begin{aligned}
\hat{Y}_t = Y_L =& \frac{\beta\pi_b(\mu-1)b_L}{\kappa} - \frac{(1-\beta\mu)\left(\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)\right)}{\kappa(\kappa\mu\sigma - (1-\mu)(1-\beta\mu))}, \\
\hat{\pi}_t = \pi_L =& -\frac{\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)}{\kappa\mu\sigma - (1-\mu)(1-\beta\mu)}, \\
\hat{\imath}_t = i_L =& 1-\beta^{-1},\ \hat{b}_t = b_L, \\
\hat{S}_t = S_L =& \frac{\rho\beta(1-\mu)S_b b_L - \hat{\imath}_t}{1-\rho\beta\mu}
\end{aligned}$$

where the solution for variables with a b subscript has already been provided above.

8.7 Non-linear markov equilibrium of extended model

We now consider a model where the duration of government debt is time-varying and chosen optimally by the government.

The policy problem can be written as

$$J\left(b_{t-1}, \rho_{t-1}, \xi_t\right) = \max\left[U\left(\Lambda_t, \xi_t\right) + \beta E_t J\left(b_t, \rho_t, \xi_{t+1}\right)\right]$$

st

$$S_t(\rho_t)b_t = \left(1 + \rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-1} + \left(F - T_t\right).$$

$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e}$$

$$i_t \geq 0$$

$$S_t(\rho_t) = \frac{1}{u_C\left(C_t, \xi_t\right)}\beta g_t^e$$

$$W_t(\rho_{t-1}) = \frac{1}{u_C\left(C_t, \xi_t\right)}\beta j_t^e$$

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)\,u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right)d'\left(\Pi_t\right)\Pi_t = \beta h_t^e$$

$$Y_t = C_t + F + d\left(\Pi_t\right)$$

$$f_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{f}^e\left(b_t, \rho_t, \xi_t\right)$$

$$g_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1 + \rho_t S_{t+1}(\rho_t)\right)\right] = \bar{g}^e\left(b_t, \rho_t, \xi_t\right)$$

$$h_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] = \bar{h}^e\left(b_t, \rho_t, \xi_t\right)$$

$$j_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1 + \rho_{t-1}W_{t+1}(\rho_{t-1})\right)\right] = \bar{j}^e\left(b_t, \rho_t, \xi_t\right)$$

Formulate the period Lagrangian

$$\begin{aligned}
L_t = {} & u\left(C_t,\xi_t\right) + g\left(F - s(T_t - T)\right) - \tilde{v}\left(Y_t\right) + \beta E_t J\left(b_t,\rho_t,\xi_{t+1}\right) \\
& + \phi_{1t}\left(S_t(\rho_t)b_t - \left(1+\rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-1} - (F - T_t)\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t,\xi_t\right)}{1+i_t}\right) \\
& + \phi_{3t}\left(\beta g_t^e - u_C\left(C_t,\xi_t\right)S_t(\rho_t)\right) \\
& + \phi_{4t}\left(\beta j_t^e - u_C\left(C_t,\xi_t\right)W_t(\rho_{t-1})\right) \\
& + \phi_{5t}\left(\beta h_t^e - \varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)\right] - u_C\left(C_t,\xi_t\right)d'\left(\Pi_t\right)\Pi_t\right) \\
& + \phi_{6t}\left(Y_t - C_t - F - d\left(\Pi_t\right)\right) \\
& + \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{2t}\left(g_t^e - \bar{g}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{3t}\left(h_t^e - \bar{h}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{4t}\left(j_t^e - \bar{j}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}$$

First-order conditions (where all the derivatives should be equated to zero)

$$\frac{\partial L_s}{\partial \Pi_t} = \phi_{1t}\left[\left(1+\rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-2}\right] + \phi_{5t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{6t}\left[-d'\right]$$

$$\frac{\partial L_s}{\partial Y_t} = -\tilde{v}_Y + \phi_{5t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t\tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{6t}$$

$$\frac{\partial L_s}{\partial i_t} = \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t}$$

$$\frac{\partial L_s}{\partial S_t} = \phi_{1t}\left[b_t\right] + \phi_{3t}\left[-u_C\right]$$

$$\frac{\partial L_s}{\partial W_t} = \phi_{1t}\left[\rho_{t-1}b_{t-1}\Pi_t^{-1}\right] + \phi_{4t}\left[-u_C\right]$$

$$\frac{\partial L_s}{\partial C_t} = u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho_t)\right] + \phi_{4t}\left[-u_{CC}W_t(\rho_{t-1})\right] + \phi_{5t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{6t}\left[-1\right]$$

$$\frac{\partial L_s}{\partial T_t} = g_G\left(-s'(T_t - T)\right) + \phi_{1t}$$

$$\frac{\partial L_s}{\partial b_t} = \beta E_t J_b\left(b_t,\rho_t,\xi_{t+1}\right) + \phi_{1t}\left[S_t(\rho_t)\right] + \psi_{1t}\left[-\bar{f}_b^e\right] + \psi_{2t}\left[-\bar{g}_b^e\right] + \psi_{3t}\left[-\bar{h}_b^e\right] + \psi_{4t}\left(-\bar{j}_b^e\right)$$

$$\frac{\partial L_s}{\partial \rho_t} = \beta E_t J_\rho\left(b_t,\rho_t,\xi_{t+1}\right)$$

$$\frac{\partial L_s}{\partial f_t^e} = \beta\phi_{2t} + \psi_{1t}$$

$$\frac{\partial L_s}{\partial g_t^e} = \beta\phi_{3t} + \psi_{2t}$$

$$\frac{\partial L_s}{\partial j_t^e} = \beta\phi_{4t} + \psi_{4t}$$

$$\frac{\partial L_s}{\partial h_t^e} = \beta\phi_{5t} + \psi_{3t}$$

The complementary slackness conditions are

$$\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t}i_t = 0$$

While the envelope conditions are

$$J_b\left(b_{t-1},\rho_{t-1},\xi_t\right) = \phi_{1t}\left[-\left(1+\rho_{t-1}W_t(\rho_{t-1})\right)\Pi_t^{-1}\right]$$

$$J_\rho\left(b_{t-1},\rho_{t-1},\xi_t\right) = \phi_{1t}\left[-W_t(\rho_{t-1})b_{t-1}\Pi_t^{-1}\right].$$